12 Analysis of Indeterminate Beams and Frames

12.1 Continuity

The individual members that compose a steel or timber structure are fabricated or cut separately and joined together by rivets, bolts, welds, or nails. Unless the joints are specially designed for rigidity, they are too flexible to transfer moments of significant magnitude from one member to another. In contrast, in reinforced concrete structures, as much of the concrete as is practical is placed in one single operation. Reinforcing steel is not terminated at the ends of a member but is extended through the joints into adjacent members. At construction joints, special care is taken to bond the new concrete to the old by carefully cleaning the latter, by extending the reinforcement through the joint, and by other means. As a result, reinforced concrete structures usually represent monolithic, or continuous, units. A load applied at one location causes deformation and stress at all other locations. Even in precast concrete construction, which resembles steel construction in that individual members are brought to the job site and joined in the field, connections are often designed to provide for the transfer of moment as well as shear and axial load, producing at least partial continuity.

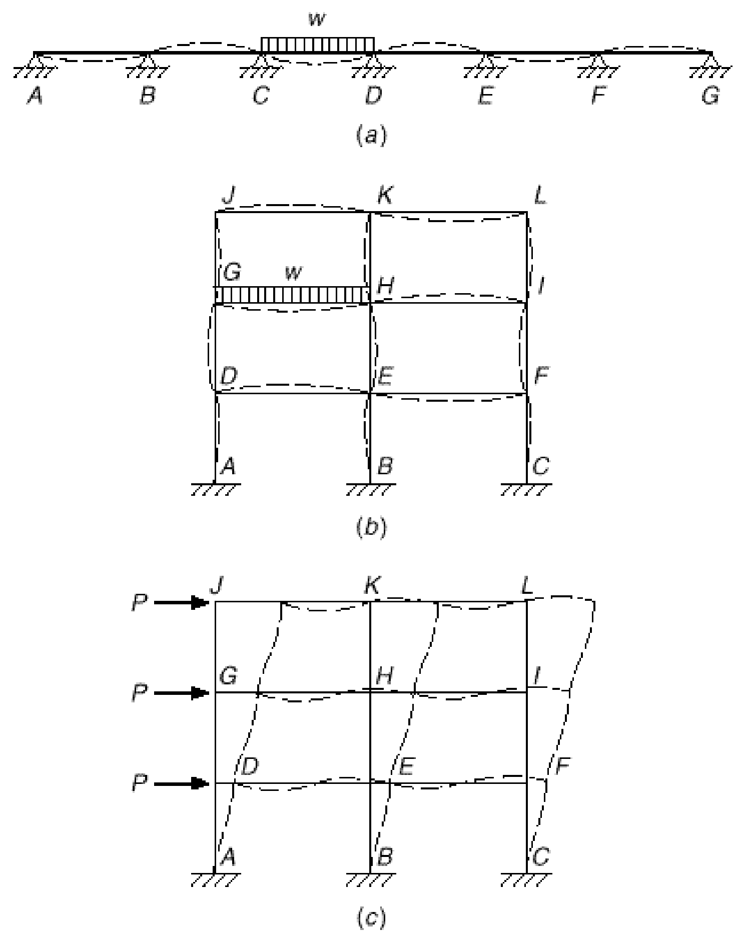
The effect of continuity is most simply illustrated by a continuous beam, such as shown in Fig. 12.1*a*. With simple spans, such as provided in many types of steel construction, only the loaded member *CD* would deform, and all other members of the structure would remain straight. But with continuity from one member to the next through the support regions, as in a reinforced concrete structure, the distortion caused by a load on one single span is seen to spread to all other spans, although the magnitude of deformation decreases with increasing distance from the loaded member. All members of the six-span structure are subject to curvature, and thus also to bending moment, as a result of loading span *CD*.

Similarly, for the rigid-jointed frame of Fig. 12.1*b*, the distortion caused by a load on the single member *GH* spreads to all beams and all columns, although, as before, the effect decreases with increasing distance from the load. All members are subject to bending moment, even though they may carry no transverse load.

If horizontal forces, such as forces caused by wind or seismic action, act on a frame, it deforms as illustrated by Fig. 12.1*c*. Here, too, all members of the frame distort, even though the forces act only on the left side; the amount of distortion is seen to be the same for all corresponding members, regardless of their distance from the points of loading, in contrast to the case of vertical loading. A member such as *EH*, even without a directly applied transverse load, will experience deformations and associated bending moment.

In *statically determinate structures*, such as simple-span beams, the deflected shape and the moments and shears depend only on the type and magnitude of the loads

FIGURE 12.1
Deflected shape of continuous beams and frames.



and the dimensions of the member. In contrast, inspection of the *statically indeterminate structures* in Fig. 12.1 shows that the deflection curve of any member depends not only on the loads but also on the joint rotations, whose magnitudes in turn depend on the distortion of adjacent, rigidly connected members. For a rigid joint such as joint *H* in the frame shown in Fig. 12.1*b* or *c*, all the rotations at the near ends of all members framing into that joint must be the same. For a correct design of continuous beams and frames, it is evidently necessary to determine moments, shears, and thrusts considering the effect of continuity at the joints.

The determination of these internal forces in continuously reinforced concrete structures is usually based on *elastic analysis* of the structure at factored loads with methods that will be described in Sections 12.2 through 12.5. Such analysis requires knowledge of the cross-sectional dimensions of the members. Member dimensions are initially estimated during preliminary design, which is described in Section 12.6 along with guidelines for establishing member proportions. For checking the results of more exact analysis, the approximate methods of Section 12.7 are useful. For many structures, a full elastic analysis is not justified, and the ACI coefficient method of analysis described in Section 12.8 provides an adequate basis for design moments and shears.

Before failure, reinforced concrete sections are usually capable of considerable inelastic rotation at nearly constant moment, as was described in Section 6.9. This permits a *redistribution of elastic moments* and provides the basis for *plastic analysis* of beams, frames, and slabs. Plastic analysis will be developed in Section 12.9 for beams and frames and in Chapters 14 and 15 for slabs.

12.2 LOADING

The individual members of a structural frame must be designed for the worst combination of loads that can reasonably be expected to occur during its useful life. Internal moments, shears, and thrusts are brought about by the combined effect of dead and live loads, plus other loads, such as wind and earthquake, as discussed in Section 1.7. While dead loads are constant, live loads such as floor loads from human occupancy can be placed in various ways, some of which will result in larger effects than others. In addition, the various combinations of factored loads specified in Table 1.2 must be used to determine the load cases that govern member design. The subject of load placement will be addressed first.

a. Placement of Loads

In Fig. 12.2*a* only span *CD* is loaded by live load. The distortions of the various frame members are seen to be largest in, and immediately adjacent to, the loaded span and to decrease rapidly with increasing distance from the load. Since bending moments are proportional to curvatures, the moments in more remote members are correspondingly smaller than those in, or close to, the loaded span. However, the loading shown

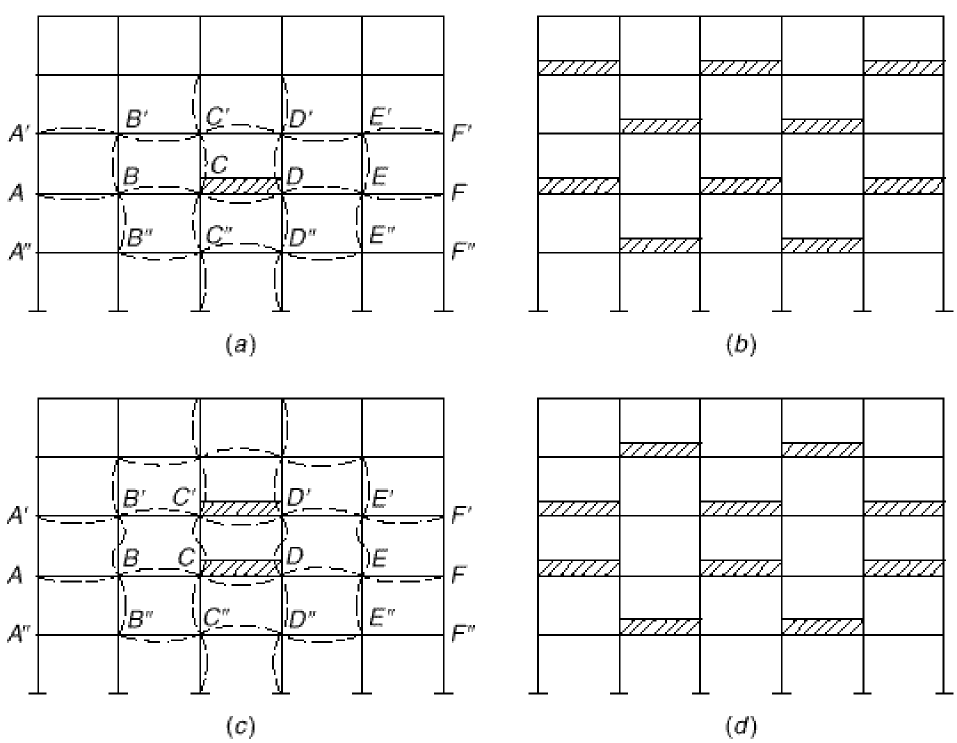


FIGURE 12.2
Alternate live loadings for maximum effects.

in Fig. 12.2*a* does not produce the maximum possible positive moment in *CD*. In fact, if additional live load were placed on span *AB*, this span would bend down, *BC* would bend up, and *CD* itself would bend down in the same manner, although to a lesser degree, as it is bent by its own load. Hence, the positive moment in *CD* is increased if *AB* and, by the same reasoning, *EF* are loaded simultaneously. By expanding the same reasoning to the other members of the frame, one can easily see that the checkerboard pattern of live load shown in Fig. 12.2*b* produces the largest possible positive moments, not only in *CD*, but in all loaded spans. Hence, two such checkerboard patterns are required to obtain the maximum positive moments in all spans.

In addition to maximum span moments, it is often necessary to investigate minimum span moments. Dead load, acting as it does on all spans, usually produces only positive span moments. However, live load, placed as in Fig. 12.2*a*, and even more so in Fig. 12.2*b*, is seen to bend the unloaded spans upward, i.e., to produce negative moments in the span. If these negative live load moments are larger than the generally positive dead load moments, a given girder, depending on load position, may be subject at one time to positive span moments and at another to negative span moments. It must be designed to withstand both types of moments; i.e., it must be furnished with tensile steel at both top and bottom. Thus, the loading of Fig. 12.2*b*, in addition to giving maximum span moments in the loaded spans, gives minimum span moments in the unloaded spans.

Maximum negative moments at the supports of the girders are obtained, on the other hand, if loads are placed on the two spans adjacent to the particular support and in a corresponding pattern on the more remote girders. A separate loading scheme of this type is then required for each support for which maximum negative moments are to be computed.

In each column, the largest moments occur at the top or bottom. While the loading shown in Fig. 12.2*c* results in large moments at the ends of columns CC' and DD', the reader can easily be convinced that these moments are further augmented if additional loads are placed as shown in Fig. 12.2*d*.

It is seen from this brief discussion that, to calculate the maximum possible moments at all critical points of a frame, live load must be placed in a great variety of different schemes. In most practical cases, however, consideration of the relative magnitude of effects will permit limitation of analysis to a small number of significant cases.

b. Load Combinations

The ACI Code requires that structures be designed for a number of load combinations, as discussed in Section 1.7. Thus, for example, factored load combinations might include (1) dead plus live load, (2) dead plus fluid plus temperature plus live plus soil plus snow load, and (3) three possible combinations that include dead, live, and wind load, with some of the combinations including snow, rain, soil, and roof live load. While each of the combinations may be considered as an individual loading condition, experience has shown that the most efficient technique involves separate analyses for each of the basic loads without load factors, that is, a full analysis for unfactored dead load only, separate analyses for the various live load distributions described in Section 12.2*a*, and separate analyses for each of the other loads (wind, snow, etc.). Once the separate analyses are completed, it is a simple matter to combine the results using the appropriate load factor for each type of load. This procedure is most advantageous because, for example, live load may require a load factor of 1.6 for one combination, a value of 1.0 for another, and a value of 0.5 for yet another. Once the forces have been calculated for each combination, the combination of loads that governs for each member can usually be identified by inspection.

12.3 Simplifications in Frame Analysis

Considering the complexity of many practical building frames and the need to account for the possibility of alternative loadings, there is evidently a need to simplify. This can be done by means of certain approximations that allow the determination of moments with reasonable accuracy while substantially reducing the amount of computation.

Numerous trial computations have shown that, for building frames with a reasonably regular outline, not involving unusual asymmetry of loading or shape, the influence of sidesway caused by vertical loads can be neglected. In that case, moments due to vertical loads are determined with sufficient accuracy by dividing the entire frame into simpler subframes. Each of these consists of one continuous beam, plus the top and bottom columns framing into that particular beam. Placing the live loads on the beam in the most unfavorable manner permits sufficiently accurate determination of all beam moments, as well as the moments at the top ends of the bottom columns and the bottom ends of the top columns. For this partial structure, the far ends of the columns are considered fixed, except for such first-floor or basement columns where soil and foundation conditions dictate the assumption of hinged ends. Such an approach is explicitly permitted by ACI Code 8.9, which specifies the following for floor and roof members:

1. The live load may be considered to be applied only to the floor or roof under consideration, and the far ends of columns built integrally with the structure may be considered fixed.
2. The arrangement of live load may be limited to combinations of (*a*) factored dead load on all spans with full factored live load on two adjacent spans, and (*b*) factored dead load on all spans with full factored live load on alternate spans.

When investigating the maximum negative moment at any joint, negligible error will result if the joints second removed in each direction are considered to be completely fixed. Similarly, in determining maximum or minimum span moments, the joints at the far ends of the adjacent spans may be considered fixed. Thus, individual portions of a frame of many members may be investigated separately.

Figure 12.3 demonstrates the application of the ACI Code requirements for live load on a three-span subframe. The loading in Fig. 12.3*a* results in maximum positive moments in the exterior spans, the minimum positive moment in the center span, and the maximum negative moments at the interior faces of the exterior columns. The loading shown in Fig. 12.3*b* results in the maximum positive moment in the center span and minimum positive moments in the exterior spans. The loading in Fig. 12.3*c* results in maximum negative moment at both faces of the interior columns. Since the structure is symmetrical, values of moment and shear obtained for the loading shown in Fig. 12.3*c* apply to the right side of the structure as well as the left. Due to the simplicity of this structure, joints away from the spans of interest are not treated as fixed.

Moments and shears used for design are determined by combining the moment and shear diagrams for the individual load cases to obtain the maximum values along each span length. The resulting envelope moment and shear diagrams are shown in Figs. 12.3*d* and *e*, respectively. The moment and shear envelopes (note the range of

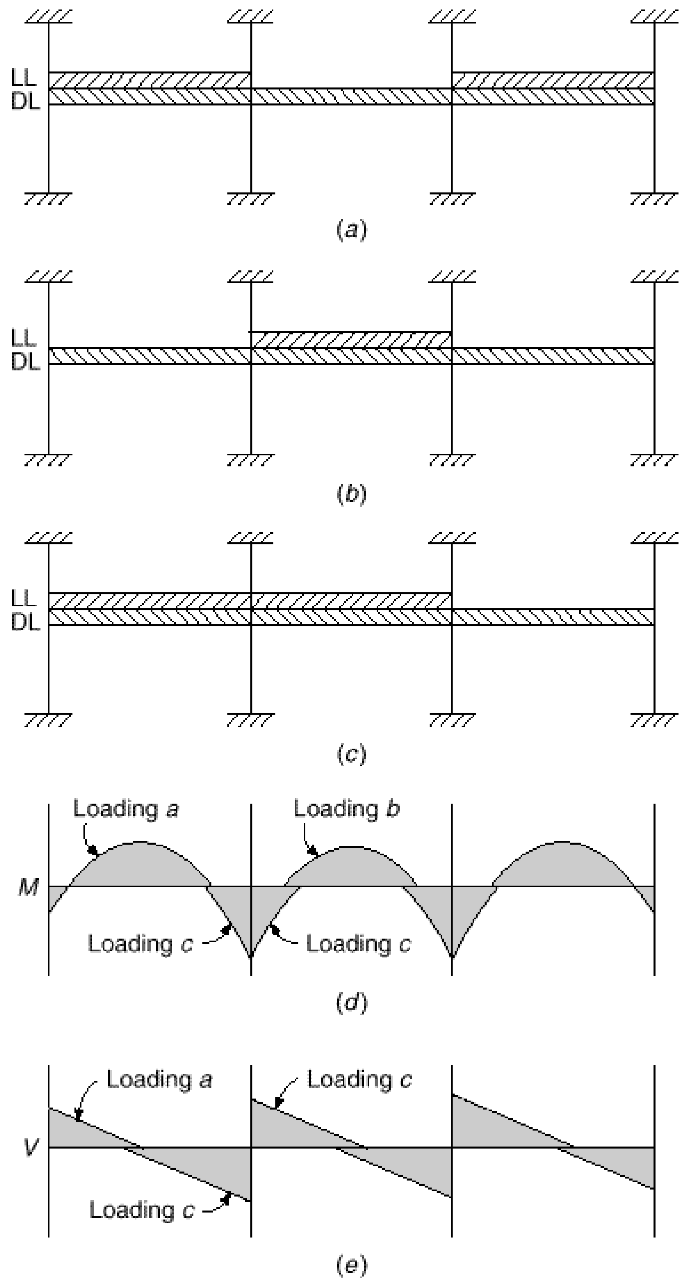


FIGURE 12.3
Subframe loading as required by ACI Code 8.9: Loading for (*a*) maximum positive moments in the exterior spans, the minimum positive moment in the center span, and the maximum negative moments at the interior faces of the exterior columns; (*b*) maximum positive moment in the center span and minimum positive moments in the exterior spans; and (*c*) maximum negative moment at both faces of the interior columns; (*d*) envelope moment diagram; (*e*) envelope shear diagram. (DL and LL represent factored dead and live loads, respectively.)

positions for points of inflection and points of zero shear) are used not only to design the critical sections but to determine cutoff points for flexural reinforcement and requirements for shear reinforcement.

In regard to columns, ACI Code 8.8 indicates:

1. Columns shall be designed to resist the axial forces from factored loads on all floors or roof and the maximum moment from factored loads on a single adjacent span of

the floor or roof under consideration. The loading condition giving the maximum ratio of moment to axial load shall also be considered.

2. In frames or continuous construction, consideration shall be given to the effect of unbalanced floor or roof loads on both exterior and interior columns and of eccentric loading due to other causes.
3. In computing moments in columns due to gravity loading, the far ends of columns built integrally with the structure may be considered fixed.
4. The resistance to moments at any floor or roof level shall be provided by distributing the moment between columns immediately above and below the given floor in proportion to the relative column stiffness and conditions of restraint.

Although it is not addressed in the ACI Code, axial loads on columns are usually determined based on the column tributary areas, which are defined based on the midspan of flexural members framing into each column. The axial load from the tributary area is used in design, with the exception of first interior columns, which are typically designed for an extra 10 percent axial load to account for the higher shear expected in the flexural members framing into the exterior face of first interior columns. The use of this procedure to determine axial loads due to gravity is conservative (note that the total vertical load exceeds the factored loads on the structure) and is adequately close to the values that would be obtained from a more detailed frame analysis.

12.4 Methods for Elastic Analysis

Many methods have been developed over the years for the elastic analysis of continuous beams and frames. The so-called classical methods (Ref. 12.1), such as application of the theorem of three moments, the method of least work (Castigliano's second theorem), and the general method of consistent deformation, will prove useful only in the analysis of continuous beams having few spans or of very simple frames. For the more complicated cases generally encountered in practice, such methods prove exceedingly tedious, and alternative approaches are preferred.

For many years moment distribution (Ref. 12.1) provided the basic analytical tool for the analysis of indeterminate concrete beams and frames, originally with the aid of the slide rule and later with handheld programmable calculators. For relatively small problems, moment distribution may still provide the most rapid results, and it is often used in current practice. However, with the widespread availability of computers, manual methods have been replaced largely by matrix analysis, which provides rapid solutions with a high degree of accuracy (Refs. 12.2 and 12.3).

Approximate methods of analysis, based either on careful sketches of the shape of the deformed structure under load or on moment coefficients, still provide a means for rapid estimation of internal forces and moments (Ref. 12.4). Such estimates are useful in preliminary design and in checking more exact solutions for gross errors that might result from input errors. In structures of minor importance, approximations may even provide the basis for final design.

In view of the number of excellent texts now available that treat methods of analysis (Refs. 12.1 to 12.4 to name just a few), the present discussion will be confined to an evaluation of the usefulness of several of the more important of these, with particular reference to the analysis of reinforced concrete structures. Certain idealizations and approximations that facilitate the solution in practical cases will be described in more detail.

a. Moment Distribution

In 1932, Hardy Cross developed the method of moment distribution to solve problems in frame analysis that involve many unknown joint displacements and rotations. For the next three decades, moment distribution provided the standard means in engineering offices for the analysis of indeterminate frames. Even now, it serves as the basic analytical tool if computer facilities are not available.

In the moment distribution method (Ref. 12.1), the fixed-end moments for each member are modified in a series of cycles, each converging on the precise final result, to account for rotation and translation of the joints. The resulting series can be terminated whenever one reaches the degree of accuracy required. After member end moments are obtained, all member stress resultants can be obtained from the laws of statics.

It has been found by comparative analyses that, except in unusual cases, building-frame moments found by modifying fixed-end moments by only two cycles of moment distribution will be sufficiently accurate for design purposes (Ref. 12.5).

b. Matrix Analysis

Use of matrix theory makes it possible to reduce the detailed numerical operations required in the analysis of an indeterminate structure to systematic processes of matrix manipulation that can be performed automatically and rapidly by computer. Such methods permit the rapid solution of problems involving large numbers of unknowns. As a consequence, less reliance is placed on special techniques limited to certain types of problems, and powerful methods of general applicability have emerged, such as the direct stiffness method (Refs. 12.2 and 12.3). By such means, an "exact" determination of internal forces throughout an entire building frame can be obtained quickly and at small expense. Three-dimensional frame analysis is possible where required. A large number of alternative loadings can be considered, including dynamic loads.

Some engineering firms prefer to write and maintain their own programs for structural analysis particularly suited to their needs. However, most make use of readily available programs that can be used for a broad range of problems. Input—including loads, material properties, structural geometry, and member dimensions—is provided by the user, often in an interactive mode. Output includes joint displacements and rotations, plus moment, shear, and thrust at critical sections throughout the structure. A number of programs are available, e.g., PCA-FRAME (Portland Cement Association, Skokie, Illinois) and others from numerous private firms. Most of these programs perform analysis of two or three-dimensional framed structures subject to static or dynamic loads, shear walls, and other elements in a small fraction of the time formerly required, providing results to a high degree of accuracy. Generally, ordinary desktop computers suffice.

12.5 Idealization of the Structure

It is seldom possible for the engineer to analyze an actual complex redundant structure. Almost without exception, certain idealizations must be made in devising an analytical model, so that the analysis will be practically possible. Thus, three-dimensional members are represented by straight lines, generally coincident with the actual centroidal axis. Supports are idealized as rollers, hinges, or rigid joints. Loads actually distributed over a finite area are assumed to be point loads. In three-dimensional framed structures, analysis is often limited to plane frames, each of which is assumed to act independently.

In the idealization of reinforced concrete frames, certain questions require special comment. The most important of these pertain to effective span lengths, effective moments of inertia, and conditions of support.

a. Effective Span Length

In elastic frame analysis, a structure is usually represented by a simple line diagram, based dimensionally on the centerline distances between columns and between floor beams. Actually, the depths of beams and the widths of columns (in the plane of the frame) amount to sizable fractions of the respective lengths of these members; their clear lengths are therefore considerably smaller than their centerline distances between joints.

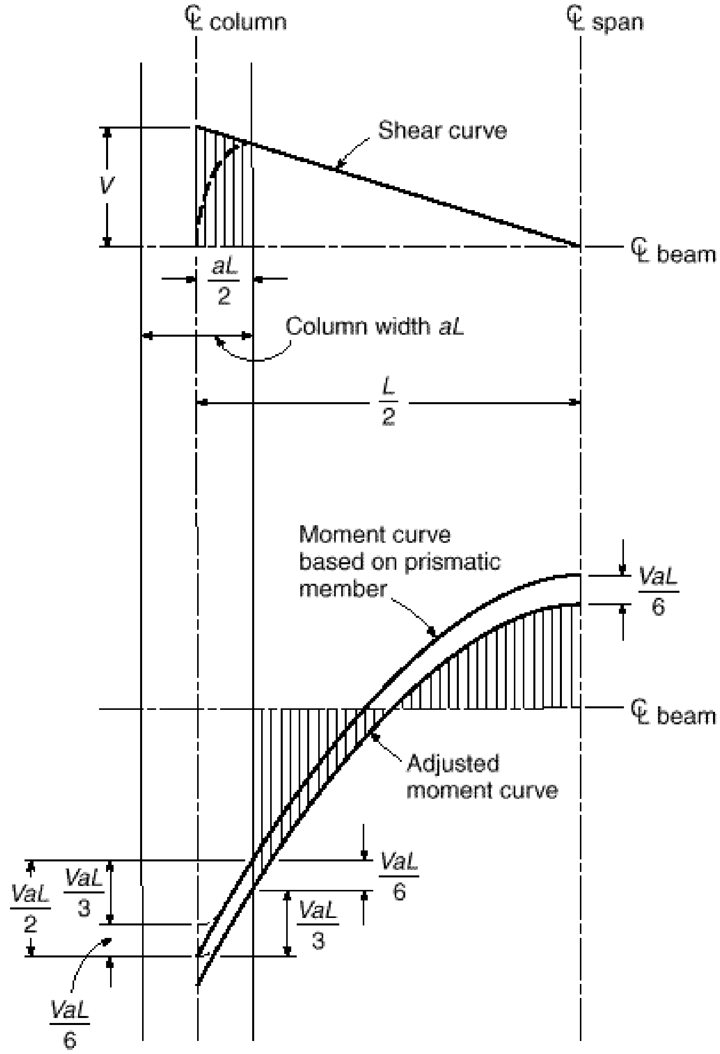
It is evident that the usual assumption in frame analysis that the members are prismatic, with constant moment of inertia between centerlines, is not strictly correct. A beam intersecting a column may be prismatic up to the column face, but from that point to the column centerline it has a greatly increased depth, with a moment of inertia that could be considered infinite compared with that of the remainder of the span. A similar variation in width and moment of inertia is obtained for the columns. Thus, to be strictly correct, the actual variation in member depth should be considered in the analysis. Qualitatively, this would increase beam support moments somewhat and decrease span moments. In addition, it is apparent that the critical section for design for negative bending would be at the face of the support, and not at the centerline, since for all practical purposes an unlimited effective depth is obtained in the beam across the width of the support.

It will be observed that, in the case of the columns, the moment gradient is not very steep, so that the difference between centerline moment and the moment at the top or bottom face of the beam is small and can in most cases be disregarded. However, the slope of the moment diagram for the beam is usually quite steep in the region of the support, and there will be a substantial difference between the support centerline moment and face moment. If the former were used in proportioning the member, an unnecessarily large section would result. It is desirable, then, to reduce support moments found by elastic analysis to account for the finite width of the supports.

In Fig. 12.4, the change in moment between the support centerline and the support face will be equal to the area under the shear diagram between those two points. For knife-edge supports, this shear area is seen to be very nearly equal to $V_aL/2$. Actually, however, the reaction is distributed in some unknown way across the width of the support. This will have the effect of modifying the shear diagram as shown by the dashed line; it has been proposed that the reduced area be taken as equal to $V_aL/3$. The fact that the reaction is distributed will modify the moment diagram as well as the shear diagram, causing a slight rounding of the negative moment peak, as shown in the figure, and the reduction of $V_aL/3$ is properly applied to the moment diagram after the peak has been rounded. This will give nearly the same face moment as would be obtained by deducting the amount $V_aL/2$ from the peak moment.

Another effect is present, however: the modification of the moment diagram due to the increased moment of inertia of the beam at the column. This effect is similar to

FIGURE 12.4
Reduction of negative and positive moments in a frame.



that of a haunch, and it will mean slightly increased negative moment and slightly decreased positive moment. For ordinary values of the ratio a, this shift in the moment curve will be on the order of $VaL/6$. Thus, it is convenient simply to deduct the amount $VaL/3$ from the unrounded peak moment obtained from elastic analysis. This allows for (1) the actual rounding of the shear diagram and the negative moment peak due to the distributed reaction and (2) the downward shift of the moment curve due to the haunch effect at the supports. The consistent reduction in positive moment of $VaL/6$ is illustrated in Fig. 12.4.

With this said, there are two other approaches that are often used by structural designers. The first is to analyze the structure based on the simple line diagram and to reduce the moment from the column centerline to the face of the support by $VaL/2$ without adjusting for the higher effective stiffness within the thickness width of the column. The moment diagram, although somewhat less realistic than represented by the lower curve in Fig. 12.4, still satisfies statics and requires less flexural reinforcement at the face of the support. As a consequence, there is less congestion in the beam-column joint location where it is often difficult to place concrete because of the high quantity of reinforcing steel from the flexural members framing into the column (usually from two different directions) and from the column itself. The somewhat higher percentage of reinforcement required at midspan usually causes little difficulty in concrete placement. The second approach involves representing the portion of the "beam" within the width of the column as a rigid link that connects the column centerline with the clear span of the flexural member. The portion of the column within the depth of the beam can also be represented using a rigid link. Such a model will produce moment diagrams similar to the lower curve in Fig. 12.4, without additional analysis. The second approach is both realistic and easy to implement in matrix analysis programs.

It should be noted that there are certain conditions of support for which no reduction in negative moment is justified. For example, when a continuous beam is carried by a girder of approximately the same depth, the negative moment in the beam at the centerline of the girder should be used to design the negative reinforcing steel.

b. Moments of Inertia

Selection of reasonable values for moments of inertia of beams and columns for use in the frame analysis is far from a simple matter. The design of beams and columns is based on cracked section theory, i.e., on the supposition that tension concrete is ineffective. It might seem, therefore, that moments of inertia to be used should be determined in the same manner, i.e., based on the cracked transformed section, in this way accounting for the effects of cracking and presence of reinforcement. Things are not this simple, unfortunately.

Consider first the influence of cracking. For typical members, the moment of inertia of a cracked beam section is about one-half that of the uncracked gross concrete section. However, the extent of cracking depends on the magnitude of the moments relative to the cracking moment. In beams, no flexural cracks would be found near the inflection points. Columns, typically, are mostly uncracked, except for those having relatively large eccentricity of loading. A fundamental question, too, is the load level to consider for the analysis. Elements that are subject to cracking will have more extensive cracks near ultimate load than at service load. Compression members will be unaffected in this respect. Thus, the relative stiffness depends on load level.

A further complication results from the fact that the effective cross section of beams varies along a span. In the positive bending region, a beam usually has a T section. For typical T beams, with flange width about 4 to 6 times web width and flange thickness from 0.2 to 0.4 times the total depth, the gross moment of inertia will be about 2 times that of the rectangular web with width b_w and depth h. However, in the negative bending region near the supports, the bottom of the section is in compression. The T flange is cracked, and the effective cross section is therefore rectangular.

The amount and arrangement of reinforcement are also influential. In beams, if bottom bars are continued through the supports, as is often done, this steel acts as compression reinforcement and stiffens the section. In columns, reinforcement ratios are generally much higher than in beams, adding to the stiffness.

Given these complications, it is clear that some simplifications are necessary. It is helpful to note that, in most cases, it is only the *ratio* of member stiffnesses that influences the final result, not the absolute value of the stiffnesses. The stiffness ratios may be but little affected by different assumptions in computing moment of inertia if there is consistency for all members.

In practice, it is generally sufficiently accurate to base stiffness calculations for frame analysis on the gross concrete cross section of the columns. In continuous T beams, cracking will reduce the moment of inertia to about one-half that of the uncracked section. Thus, the effect of the flanges and the effect of cracking may nearly cancel in the positive bending region. In the negative moment regions there are no flanges; however, if bottom bars continue through the supports to serve as compression steel, the added stiffness tends to compensate for lack of compression flange. Thus, for beams, generally a constant moment of inertia can be used, based on the rectangular cross-sectional area $b_w h$.

ACI Code 8.6.1 states that *any* set of reasonable assumptions may be used for computing relative stiffnesses, provided that the assumptions adopted are consistent throughout the analysis. ACI Commentary R8.6.1 notes that *relative* values of stiffness are important and that two common assumptions are to use gross EI values for all members or to use half the gross EI of the beam stem for beams and the gross EI for the columns. Additional guidance is given in ACI Code 10.11.1, which specifies the section properties to be used for frames subject to sidesway. Thirty-five percent of the *gross* moment of inertia is used for beams and 70 percent for columns. This differs from the guidance provided in ACI Commentary 8.6.1 but, except for a factor of 0.70, matches the guidance provided in the earlier discussion.

c. Conditions of Support

For purposes of analysis, many structures can be divided into a number of two-dimensional frames. Even for such cases, however, there are situations in which it is impossible to predict with accuracy what the conditions of restraint might be at the ends of a span; yet moments are frequently affected to a considerable degree by the choice made. In many other cases, it is necessary to recognize that structures may be three-dimensional. The rotational restraint at a joint may be influenced or even governed by the characteristics of members framing into that joint at right angles. Adjacent members or frames parallel to the one under primary consideration may likewise influence its performance.

If floor beams are cast monolithically with reinforced concrete walls (frequently the case when first-floor beams are carried on foundation walls), the moment of inertia of the wall about an axis parallel to its face may be so large that the beam end could be considered completely fixed for all practical purposes. If the wall is relatively thin or the beam particularly massive, the moment of inertia of each should be calculated, that of the wall being equal to $bt^3/12$, where t is the wall thickness and b the wall width tributary to one beam.

If the outer ends of concrete beams rest on masonry walls, as is sometimes the case, an assumption of zero rotational restraint (i.e., hinged support) is probably closest to the actual case.

For columns supported on relatively small footings, which in turn rest on compressible soil, a hinged end is generally assumed, since such soils offer but little resistance to rotation of the footing. If, on the other hand, the footings rest on solid rock, or if a cluster of piles is used with their upper portion encased by a concrete cap, the effect is to provide almost complete fixity for the supported column, and this should be assumed in the analysis. Columns supported by a continuous foundation mat should likewise be assumed fixed at their lower ends.

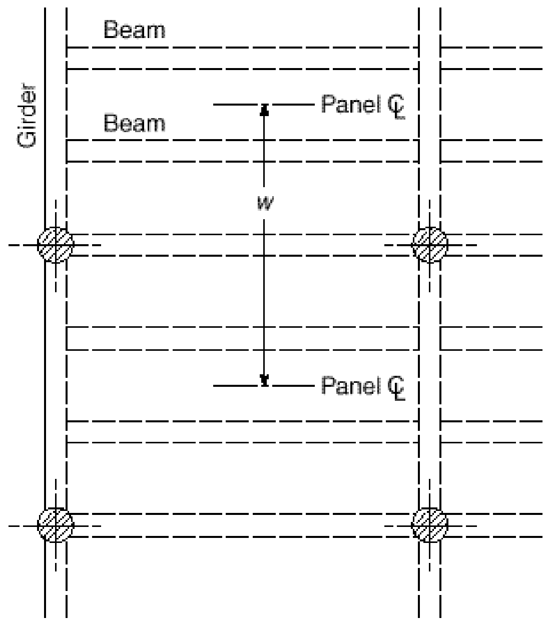


FIGURE 12.5
Slab, beam, and girder floor system.

If members framing into a joint in a direction perpendicular to the plane of the frame under analysis have sufficient torsional stiffness, and if their far ends are fixed or nearly so, their effect on joint rigidity should be included in the computations. The torsional stiffness of a member of length L is given by the expression GJ/L, where G is the shear modulus of elasticity of concrete (approximately to $E_c/2.2$) and J is the torsional stiffness factor of the member. For beams with rectangular cross sections or with sections made up of rectangular elements, J can be taken equal to $\Sigma(hb^3/3 - b^4/5)$, in which h and b are the cross-sectional dimensions of each rectangular element, b being the lesser dimension in each case. In moment distribution, when the effect of torsional rigidity is included, it is important that the absolute flexural stiffness $4EI/L$ be used rather than relative I/L values.

A common situation in beam-and-girder floors and concrete joist floors is illustrated in Fig. 12.5. The sketch shows a beam-and-girder floor system in which longitudinal beams are placed at the third points of each bay, supported by transverse girders, in addition to the longitudinal beams supported directly by the columns. If the transverse girders are quite stiff, it is apparent that the flexural stiffness of all beams in the width w should be balanced against the stiffness of one set of columns in the longitudinal bent. If, on the other hand, the girders have little torsional stiffness, there would be ample justification for making two separate longitudinal analyses, one for the beams supported directly by the columns, in which the rotational resistance of the columns would be considered, and a second for the beams framing into the girders, in which case hinged supports would be assumed. In most cases, it would be sufficiently accurate to consider the girders stiff torsionally and to add directly the stiffness of all beams tributary to a single column. This has the added advantage that all longitudinal beams will have the same cross-sectional dimensions and the same reinforcing steel, which will greatly facilitate construction. Plastic redistribution of loads upon overloading would generally ensure nearly equal restraint moments on all beams before collapse as assumed in design. Torsional moments should not be neglected in designing the girders.

12.6 PRELIMINARY DESIGN AND GUIDELINES FOR PROPORTIONING MEMBERS

In making an elastic analysis of a structural framework, it is necessary to know at the outset the cross-sectional dimensions of the members, so that moments of inertia and stiffnesses can be calculated. Yet the determination of these same cross-sectional dimensions is the precise purpose of the elastic analysis. In terms of load, the dead load on a structure is often dominated by the weight of the slab. Obviously, a preliminary estimate of member sizes must be one of the first steps in the analysis. Subsequently, with the results of the analysis at hand, members are proportioned, and the resulting dimensions compared with those previously assumed. If necessary, the assumed section properties are modified, and the analysis is repeated. Since the procedure may become quite laborious, it is obviously advantageous to make the best possible original estimate of member sizes, in the hope of avoiding repetition of the analysis.

In this connection, it is worth repeating that in the ordinary frame analysis, one is concerned with relative stiffnesses only, not the absolute stiffnesses. If, in the original estimate of member sizes, the stiffnesses of all beams and columns are overestimated or underestimated by about the same amount, correction of these estimated sizes after the first analysis will have little or no effect. Consequently, no revision of the analysis would be required. If, on the other hand, a nonuniform error in estimation is made, and relative stiffnesses differ from assumed values by more than about 30 percent, a new analysis should be made.

The experienced designer can estimate member sizes with surprising accuracy. Those with little or no experience must rely on trial calculations or arbitrary rules, modified to suit particular situations. In building frames, the depth of one-way slabs (discussed at greater length in Chapter 13) is often controlled by either deflection requirements or the negative moments at the faces of the supporting beams. Minimum depth criteria are reflected in Table 13.1, and negative moments at the face of the support can be estimated using coefficients described in Section 12.8. A practical minimum thickness of 4 in. is often used, except for joist construction meeting the requirements of ACI Code 8.11 (see Section 18.2d).

Beam sizes are usually governed by the negative moments and the shears at the supports, where their effective section is rectangular. Moments can be approximated by the fixed-end moments for the particular span, or by using the ACI moment coefficients (see Section 12.8). In most cases, shears will not differ greatly from simple beam shears. Alternatively, many designers prefer to estimate the depth of beams at about $\frac{3}{4}$ in. per foot of span, with the width equal to about one-half the depth.

For most construction, wide, relatively shallow beams and girders are preferred to obtain minimum floor depths, and using the same depth for all flexural members allows the use of simple, low-cost forming systems. Such designs can significantly reduce forming costs, while incurring only small additional costs for concrete and reinforcing steel. It is often wise to check the reinforcement ratio · based on the assumed moments to help maintain overall economy. · ≈ 0.012 in preliminary design will give · ≈ 0.01 in a final design, if a more exact analysis is used. Obviously, member dimensions are subject to modification, depending on the type and magnitude of the loads, methods of design, and material strength.

Column sizes are governed primarily by axial loads, which can be estimated quickly, although the presence of moments in the columns is cause for some increase of the area as determined by axial loads. For interior columns, in which unbalanced moments will not be large, a 10 percent increase may be sufficient, while for exterior columns, particularly for upper stories, an increase of 50 percent in area may be appro-

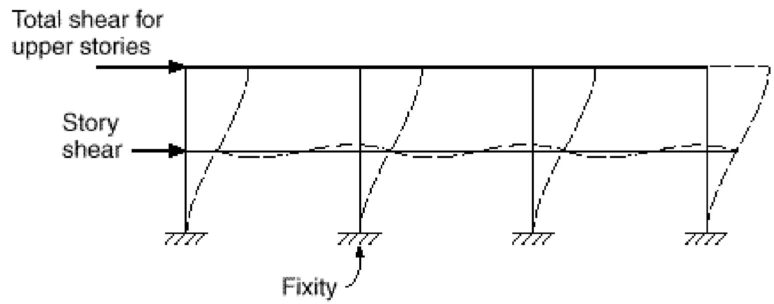


FIGURE 12.6
Subframe for estimating moments in lower-story columns of lateral load–resisting frames.

priate. In deciding on these estimated increases, the following factors should be considered. Moments are larger in exterior than in interior columns, since in the latter dead load moments from adjacent spans will largely balance, in contrast to the case in exterior columns. In addition, the influence of moments, compared with that of axial loads, is larger in upper-floor than in lower-floor columns, because the moments are usually of about the same magnitude, while the axial loads are larger in the latter than in the former.

For minimum forming costs, it is highly desirable to use the same column dimensions throughout the height of a building. This can be accomplished by using higher-strength concrete on the lower stories (for high-rise buildings, this should be the highest-strength concrete available) and reducing concrete strength in upper stories, as appropriate. For columns in *laterally braced frames*, the preliminary design of the lower-story columns may be based on zero eccentricity using $0.80 \cdot P_o = P_u$. A total reinforcement ratio $\cdot_g \approx 0.02$ should be used for the column with the highest axial load. With a value of $\cdot_g \approx 0.01$ for the column with the lowest axial load on higher stories, the column size is maintained, reducing f'_c when $\cdot_g$ drops below 1 percent. Although ACI Code 10.9.1 limits $\cdot_g$ to a range of 1 to 8 percent, the effective minimum value of $\cdot_g$ is 0.005 based on ACI Code 10.8.4, which allows the minimum reinforcement to be calculated based on a reduced effective area A_g not less than one-half the total area (this provision cannot be used in regions of high seismic risk). For columns in lateral load–resisting frames, a subframe may be used to estimate the factored bending moments due to lateral load on the lower-story columns. The subframe illustrated in Fig. 12.6 consists of the lower two stories in the structure, with the appropriate level of fixity at the base. The upper flexural members in the subframe are treated as rigid. Factored lateral loads are applied to the structure. Judicious consideration of factors such as those just discussed, along with simple models, as appropriate, will enable a designer to produce a reasonably accurate preliminary design, which in most cases will permit a satisfactory analysis to be made on the first trial.

12.7 APPROXIMATE ANALYSIS

In spite of the development of refined methods for the analysis of beams and frames, increasing attention is being paid to various approximate methods of analysis (Ref. 12.4). There are several reasons for this. Prior to performing a complete analysis of an indeterminate structure, it is necessary to estimate the proportions of its members to determine their relative stiffness, upon which the analysis depends. These dimensions can be obtained on the basis of approximate analysis. Also, even with the availability of computers, most engineers find it desirable to make a rough check of results, using approximate means, to detect gross errors. Further, for structures of minor importance, it is often satisfactory to design on the basis of results obtained by rough calculation. For these reasons, many engineers at some stage in the design process estimate the values of moments, shears, and thrusts at critical locations, using approximate sketches of the structure deflected by its loads.

Provided that points of inflection (locations in members at which the bending moment is zero and there is a reversal of curvature of the elastic curve) can be located accurately, the stress resultants for a framed structure can usually be found on the basis of static equilibrium alone. Each portion of the structure must be in equilibrium under the application of its external loads and the internal stress resultants.

For the fixed-end beam in Fig. 12.7*a*, for example, the points of inflection under uniformly distributed load are known to be located $0.211l$ from the ends of the span.

FIGURE 12.7
Analysis of fixed-end beam by locating inflection points.

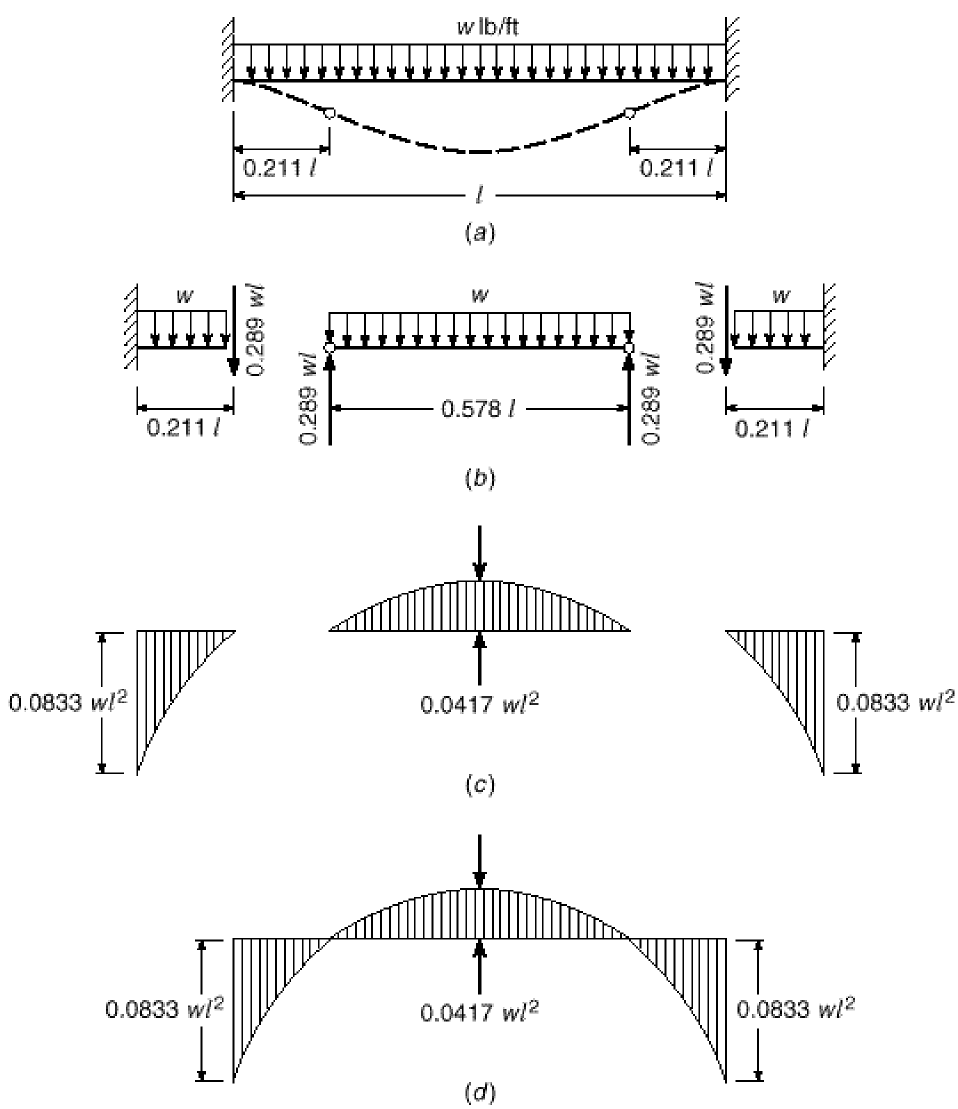
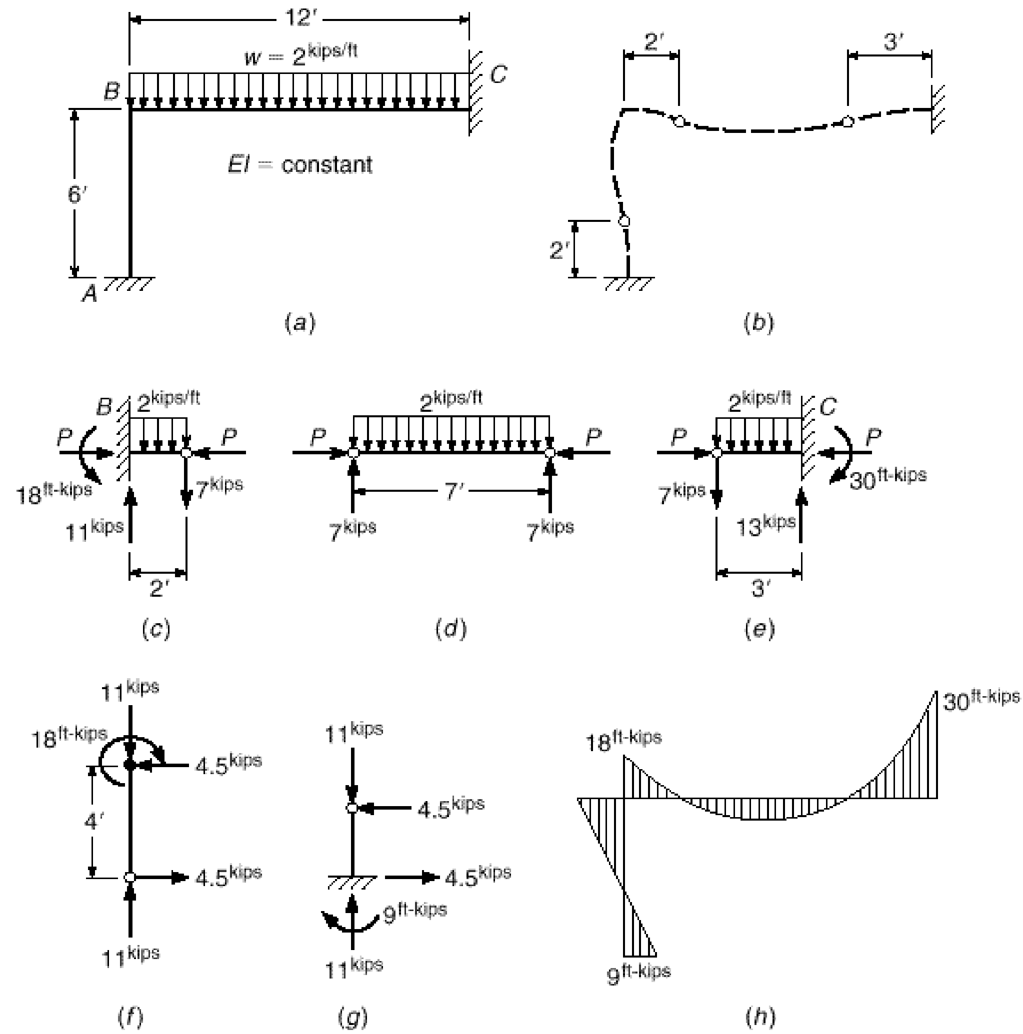


FIGURE 12.8
Approximate analysis of rigid frame.



Since the moment at these points is zero, imaginary hinges can be placed there without modifying the member behavior. The individual segments between hinges can be analyzed by statics, as shown in Fig. 12.7*b*. Starting with the center segment, shears equal to $0.289wl$ must act at the hinges. These, together with the transverse load, produce a midspan moment of $0.0417wl^2$. Proceeding next to the outer segments, a downward load is applied at the hinge representing the shear from the center segment. This, together with the applied load, produces support moments of $0.0833wl^2$. Note that, for this example, since the correct position of the inflection points was known at the start, the resulting moment diagram of Fig. 12.7*c* agrees exactly with the true moment diagram for a fixed-end beam shown in Fig. 12.7*d*. In more practical cases inflection points must be estimated, and the results obtained will only approximate the true values.

The use of approximate analysis in determining stress resultants in frames is illustrated by Fig. 12.8. Figure 12.8*a* shows the geometry and loading of a two-member rigid frame. In Fig. 12.8*b* an exaggerated sketch of the probable deflected shape is given, together with the estimated location of points of inflection. On this basis, the central portion of the girder is analyzed by statics, as shown in Fig. 12.8*d*, to obtain girder shears at the inflection points of 7 kips, acting with an axial load P (still not determined). Similarly, the requirements of statics applied to the outer segments of the girder in Fig. 12.8*c* and *e* give vertical shears of 11 and 13 kips at B and C, respectively, and end moments of 18 and 30 ft-kips at the same locations. Proceeding then to the upper segment of the column, shown in Fig. 12.8*f*, with known axial load of 11 kips and top moment of 18 ft-kips acting, a horizontal shear of 4.5 kips at the inflection point is required for equilibrium. Finally, static analysis of the lower part of the column indicates a requirement of 9 ft-kips moment at A, as shown in Fig. 12.8*g*. The value of P equal to 4.5 kips is obtained by summing horizontal forces at joint B.

The moment diagram resulting from approximate analysis is shown in Fig. 12.8*h*. For comparison, an exact analysis of the frame indicates member end moments of 8 ft-kips at A, 16 ft-kips at B, and 28 ft-kips at C. The results of the approximate analysis would be satisfactory for design in many cases; if a more exact analysis is to be made, a valuable check is available on the magnitude of results.

A specialization of the approximate method described, known as the *portal method,* is commonly used to estimate the effects of sidesway due to lateral forces acting on multistory building frames. For such frames, it is usual to assume that horizontal loads are applied at the joints only. If this is true, moments in all members vary linearly and, except in hinged members, have opposite signs close to the midpoint of each member.

For a simple rectangular portal frame having three members, the shear forces are the same in both legs and are each equal to half the external horizontal load. If one of the legs is more rigid than the other, it would require a larger horizontal force to displace it horizontally the same amount as the more flexible leg. Consequently, the portion of the total shear resisted by the stiffer column is larger than that of the more flexible column.

In multistory building frames, moments and forces in the girders and columns of each individual story are distributed in substantially the same manner as just discussed for single-story frames. The portal method of computing approximate moments, shears, and axial forces from horizontal loads is, therefore, based on the following three simple propositions:

1. The total horizontal shear in all columns of a given story is equal and opposite to the sum of all horizontal loads acting above that story.
2. The horizontal shear is the same in both exterior columns; the horizontal shear in each interior column is twice that in an exterior column.
3. The inflection points of all members, columns and girders, are located midway between joints.

Although the last of these propositions is commonly applied to all columns, including those of the bottom floor, the authors prefer to deal with the latter separately, depending on conditions of foundation. If the actual conditions are such as practically to prevent rotation (foundation on rock, massive pile foundations, etc.), the inflection points of the bottom columns are above midpoint and may be assumed to be at a distance $2h/3$ from the bottom. If little resistance is offered to rotation, e.g., for relatively small footings on compressible soil, the inflection point is located closer to the bottom and may be assumed to be at a distance $h/3$ from the bottom, or even lower. (With ideal hinges, the inflection point is at the hinge, i.e., at the very bottom.) Since shears and corresponding moments are largest in the bottom story, a judicious evaluation of

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foundation conditions as they affect the location of inflection points is of considerable importance.

The first of the three cited propositions follows from the requirement that horizontal forces be in equilibrium at any level. The second takes account of the fact that in building frames interior columns are generally more rigid than exterior ones because (1) the larger axial loads require a larger cross section and (2) exterior columns are restrained from joint rotation only by one abutting girder, while interior columns are so restrained by two such members. The third proposition is very nearly true because, except for the top and bottom columns and, to a minor degree, for the exterior girders, each member in a building frame is restrained about equally at both ends. For this reason, members deflect under horizontal loads in an antisymmetrical manner, with the inflection point at midlength.

The actual computations in this method are extremely simple. Once column shears are determined from propositions 1 and 2 and inflection points located from proposition 3, all moments, shears, and forces are simply computed by statics. The process is illustrated in Fig. 12.9*a*.

Consider joints *C* and *D*. The total shear in the second story is $3 + 6 = 9$ kips. According to proposition 2, the shear in each exterior column is $9/6 = 1.5$ kips, and in each interior column $2 \times 1.5 = 3.0$ kips. The shears in the other floors, obtained in the same manner, act at the hinges as shown. Consider the equilibrium of the rigid structure between hinges *a, b,* and *c*; the column moments, 3.0 and 9.0 ft-kips, respectively, are obtained directly by multiplying the shears by their lever arms, 6 ft. The girder moment at *C,* to produce equilibrium, is equal and opposite to the sum of the column moments. The shear in the girder is obtained by recognizing that its moment (i.e., shear times half the girder span) must be equal to the girder moment at *C*. Hence, this shear is $12.0/10 = 1.2$ kips. The moment at end *D* is equal to that at *C,* since the inflection point is at midspan. At *D,* column moments are computed in the same manner from the known column shears and lever arms. The sum of the two girder moments, to produce equilibrium, must be equal and opposite to the sum of the two column moments, from which the girder moment to the right of *C* is $18.0 + 6.0 - 12.0 = 12.0$ ft-kips. Axial forces in the columns also follow from statics. Thus, for the rigid body *aEd,* a vertical shear of 0.3 kip is seen to act upward at *d.* To equilibrate it, a tensile force of -0.3 kip is required in the column *CE.* In the rigid body *abc*, an upward shear of 1.2 kips at *b* is added to the previous upward tension of 0.3 kip at *a.* To equilibrate these two forces, a tension force of -1.5 kips is required in column *AC.* If the equilibrium of all other partial structures between hinges is considered in a similar manner, all moments, forces, and shears are rapidly determined.

In the present case, relatively flexible foundations were assumed, and the location of the lowermost inflection points was estimated to be at $h/3$ from the bottom. The general character of the resulting moment distribution is shown in Fig. 12.9*b*.

12.8 ACI Moment Coefficients

ACI Code 8.3 includes expressions that may be used for the approximate calculation of maximum moments and shears in continuous beams and one-way slabs. The expressions for moment take the form of a coefficient multiplied by $w_u l_n^2$, where w_u is the total factored load per unit length on the span and l_n is the clear span from face to face of supports for positive moment, or the average of the two adjacent clear spans for negative moment. Shear is taken equal to a coefficient multiplied by $w_u l_n/2$. The

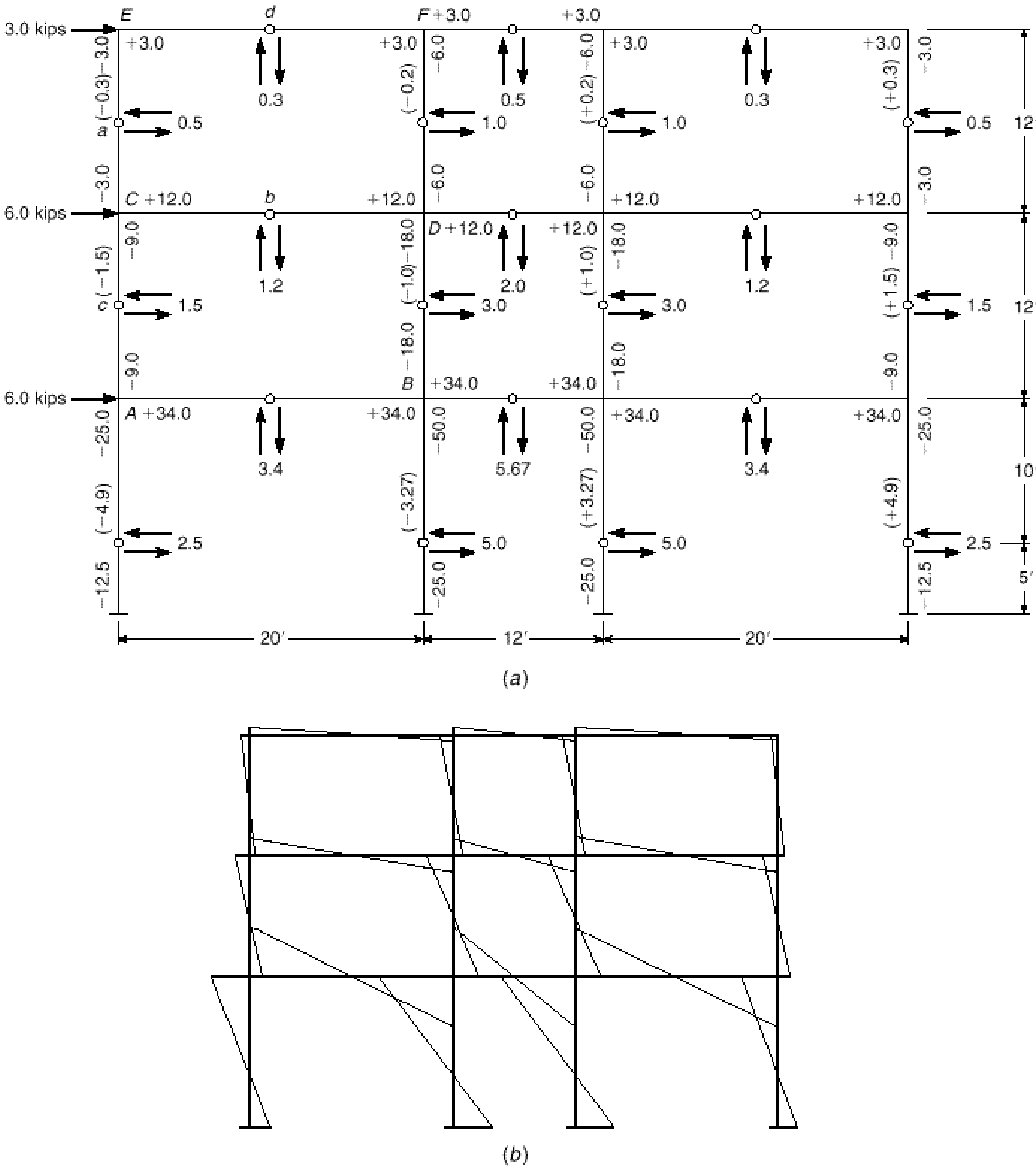


FIGURE 12.9
Portal method for determining moments from wind load in a building frame: (*a*) moments, shears, and thrusts; (*b*) variations of moments.

TABLE 12.1
Moment and shear values using ACI coefficients†

Positive moment	
End spans	
If discontinuous end is unrestrained	$\frac{1}{11} w_u l_n^2$
If discontinuous end is integral with the support	$\frac{1}{14} w_u l_n^2$
Interior spans	$\frac{1}{16} w_u l_n^2$
Negative moment at exterior face of first interior support	
Two spans	$\frac{1}{9} w_u l_n^2$
More than two spans	$\frac{1}{10} w_u l_n^2$
Negative moment at other faces of interior supports	$\frac{1}{11} w_u l_n^2$
Negative moment at face of all supports for (1) slabs with spans not exceeding 10 ft and (2) beams and girders where ratio of sum of column stiffness to beam stiffness exceeds 8 at each end of the span	$\frac{1}{12} w_u l_n^2$
Negative moment at interior faces of exterior supports for members built integrally with their supports	
Where the support is a spandrel beam or girder	$\frac{1}{24} w_u l_n^2$
Where the support is a column	$\frac{1}{16} w_u l_n^2$
Shear in end members at first interior support	$1.15 \frac{w_u l_n}{2}$
Shear at all other supports	$\frac{w_u l_n}{2}$

† w_u = total factored load per unit length of beam or per unit area of slab.
l_n = clear span for positive moment and shear and the average of the two adjacent clear spans for negative moment.

coefficients, found in ACI Code 8.3.3, are reprinted in Table 12.1 and summarized in Fig. 12.10.

The ACI moment coefficients were derived by elastic analysis, considering alternative placement of live load to yield maximum negative or positive moments at the critical sections, as was described in Section 12.2. They are applicable within the following limitations:

1. There are two or more spans.
2. Spans are approximately equal, with the longer of two adjacent spans not greater than the shorter by more than 20 percent.
3. Loads are uniformly distributed.
4. The unit live load does not exceed 3 times the unit dead load.
5. Members are prismatic.

As discussed in Section 12.3 for more general loading conditions, the alternative loading patterns considered in applying the Code moment coefficients result in an envelope of maximum moments, as illustrated in Fig. 12.11 for one span of a continuous frame. For maximum positive moment, that span would carry dead and live loads, while adjacent spans would carry dead load only, producing the diagram of Fig. 12.11*a*. For maximum negative moment at the left support, dead and live loads would

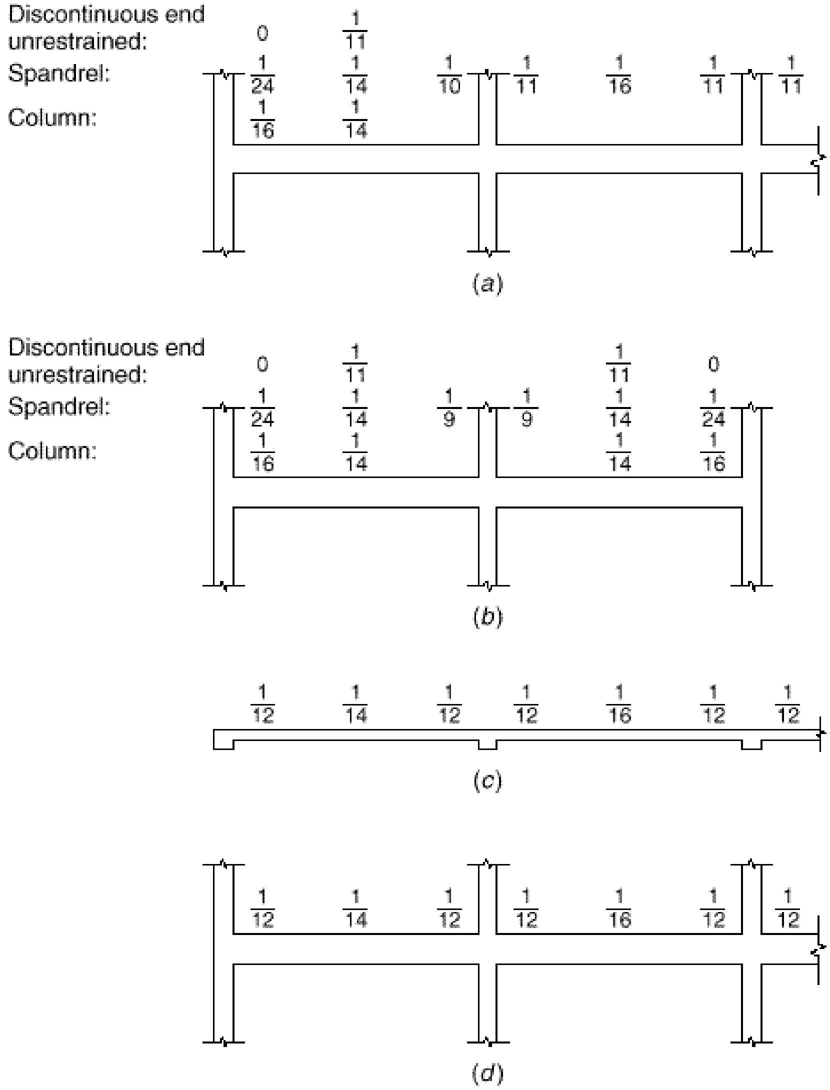


FIGURE 12.10
Summary of ACI moment coefficients: (*a*) beams with more than two spans; (*b*) beams with two spans only; (*c*) slabs with spans not exceeding 10 ft; (*d*) beams in which the sum of column stiffnesses exceeds 8 times the sum of beam stiffnesses at each end of the span.

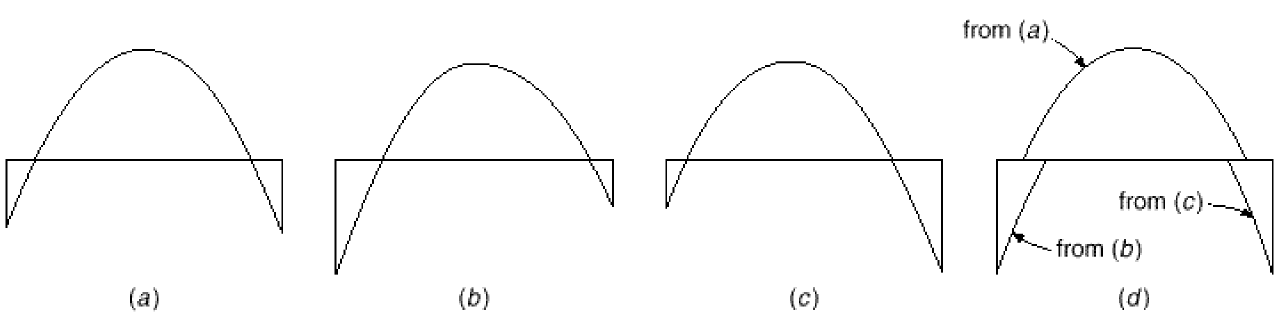


FIGURE 12.11
Maximum moment diagrams and moment envelope for a continuous beam: (*a*) maximum positive moment; (*b*) maximum negative moment at left end; (*c*) maximum negative moment at right end; (*d*) composite moment envelope.

be placed on the given span and that to the left, while the adjacent span on the right would carry only dead load, with the result shown in Fig. 12.11*b*. Figure 12.11*c* shows the corresponding results for maximum moment at the right support.

The composite moment diagram formed from the controlling portions of those just developed (Fig. 12.11*d*) provides the basis for design of the span. As observed in Section 12.3, there is a range of positions for the points of inflection resulting from alternate loadings. The extreme locations, required to determine bar cutoff points, can be found with the aid of Graph A.3 of Appendix A. In the region of the inflection point, it is evident from Fig. 12.11*d* that there may be a reversal of moments for alternative load patterns. However, within the stated limits for use of the coefficients, there should be no reversal of moments at the critical design sections near midspan or at the support faces.

Comparison of the moments found using the ACI coefficients with those calculated by more exact analysis will usually indicate that the coefficient moments are quite conservative. Actual elastic moments may be considerably smaller. Consequently, in many reinforced concrete structures, significant economy can be achieved by making a more precise analysis. This is mandatory for beams and slabs with spans differing by more than 20 percent, sustaining loads that are not uniformly distributed, or carrying live loads greater than 3 times the dead load.

Because the load patterns in a continuous frame that produce critical moments in the columns are different from those for maximum negative moments in the beams, column moments must be found separately. According to ACI Code 8.8, columns must be designed to resist the axial load from factored dead and live loads on all floors above and on the roof plus the maximum moment from factored loads on a single adjacent span of the floor or roof under consideration. In addition, because of the characteristic shape of the column strength interaction diagram (see Chapter 8), it is necessary to consider the case that gives the maximum ratio of moment to axial load. In multistory structures, this results from a checkerboard loading pattern (see Fig. 12.2*d*), which gives maximum column moments but at a less-than-maximum axial force. As a simplification, in computing moments resulting from gravity loads, the far ends of the columns may be considered fixed. The moment found at a column-beam joint for a given loading is to be assigned to the column above and the column below in proportion to the relative column stiffness and conditions of restraint.

The shears at the ends of the spans in a continuous frame are modified from the value of $w_u l_n/2$ for a simply supported beam because of the usually unbalanced end moments. For interior spans, within the limits of the ACI coefficient method, this effect will seldom exceed about 8 percent, and it may be neglected, as suggested in Table 12.1. However, for end spans, at the face of the first interior support, the additional shear is significant, and a 15 percent increase above the simple beam shear is indicated in Table 12.1. The corresponding reduction in shear at the face of the exterior support is conservatively neglected.

12.9 LIMIT ANALYSIS

a. Introduction

Most reinforced concrete structures are designed for moments, shears, and axial forces found by elastic theory with methods such as those described in Sections 12.1 through 12.8. On the other hand, the actual proportioning of members is done by strength methods, with the recognition that inelastic section and member response would result upon overloading. Factored loads are used in the elastic analysis to find moments in a continuous beam, for example, after which the critical beam sections are designed with the knowledge that the steel would be well into the yield range and the concrete stress distribution very nonlinear before final collapse. Clearly this is an inconsistent approach to the total analysis-design process, although it can be shown to be both safe and conservative. A beam or frame so analyzed and designed will not fail at a load lower than the value calculated in this way.[†]

On the other hand, it is known that a continuous beam or frame normally will not fail when the nominal moment capacity of just one critical section is reached. A *plastic hinge* will form at that section, permitting large rotation to occur at essentially constant resisting moment and thus transferring load to other locations along the span where the limiting resistance has not yet been reached. Normally in a continuous beam or frame, excess capacity will exist at those other locations because they would have been reinforced for moments resulting from different load distributions selected to produce maximum moments at those other locations.

As loading is further increased, additional plastic hinges may form at other locations along the span and eventually result in collapse of the structure, but only after a significant *redistribution of moments* has occurred. The ratio of negative to positive moments found from elastic analysis is no longer correct, for example, and the true ratio after redistribution depends upon the flexural strengths actually provided at the hinging sections.

Recognition of redistribution of moments can be important because it permits a more realistic appraisal of the actual load-carrying capacity of a structure, thus leading to improved economy. In addition, it permits the designer to modify, within limits, the moment diagrams for which members are to be designed. Certain sections can be deliberately underreinforced if moment resistance at adjacent critical sections is increased correspondingly. Adjustment of design moments in this way enables the designer to reduce the congestion of reinforcement that often occurs in high-moment areas, such as at the beam-column joints.

The formation of plastic hinges is well established by tests such as that pictured in Fig. 12.12. The three-span continuous beam illustrates the inelastic response typical of heavily overloaded members. It was reinforced in such a way that plastic hinges would form at the interior support sections before the limit capacity of sections elsewhere was reached. The beam continued to carry increasing load well beyond the load that produced first yielding at the supports. The extreme deflections and sharp changes in slope of the member axis that are seen here were obtained only slightly before final collapse.

The *inconsistency* of the present approach to the total analysis-design process, the possibility of using the *reserve strength* of concrete structures resulting from moment redistribution, and the opportunity to *reduce steel congestion* in critical regions have motivated considerable interest in limit analysis for reinforced concrete based on the concepts just described. For beams and frames, ACI Code 8.4 permits limited redistribution of moments, depending upon the strain in the tensile steel ϵ_t. For slabs, which generally use very low reinforcement ratios and consequently have great ductility, plastic design methods are especially suitable.

[†] See the discussion of upper and lower bound theorems of the theory of plasticity, Section 14.2, for an elaboration on this point.

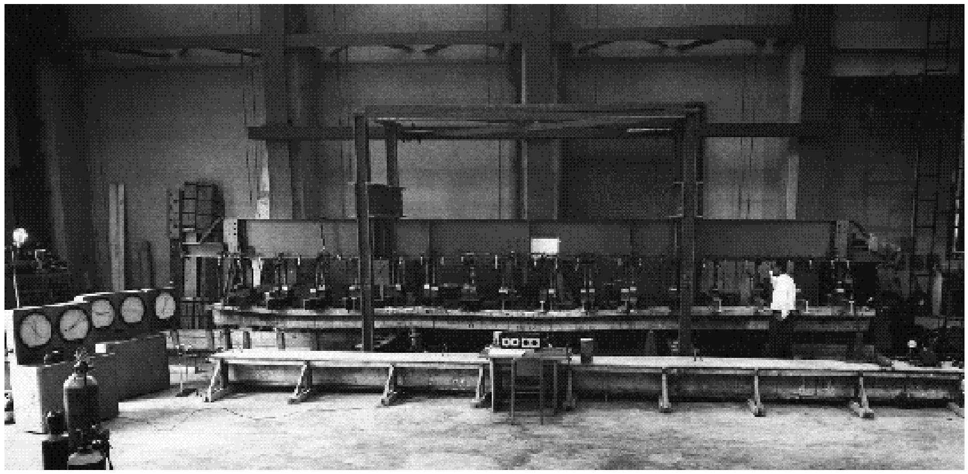


FIGURE 12.12
Three-span continuous beam after the formation of plastic hinges at the interior supports.

b. Plastic Hinges and Collapse Mechanisms

If a short segment of a reinforced concrete beam is subjected to a bending moment, curvature of the beam axis will result, and there will be a corresponding rotation of one face of the segment with respect to the other. It is convenient to express this in terms of an angular change per unit length of the member. The relation between moment and angle change per unit length of beam, or curvature, at a reinforced concrete beam section subject to tensile cracking was developed in Section 6.9. Methods were presented there by which the theoretical moment-curvature graph might be drawn for a given beam cross section, as in Fig. 6.16.

The actual moment-curvature relationship measured in beam tests differs somewhat from that shown in Fig. 6.16, mainly because, from tests, curvatures are calculated from average strains measured over a finite gage length, usually about equal to the effective depth of the beam. In particular, the sharp increase in curvature upon concrete cracking shown in Fig. 6.16 is not often seen because the crack occurs at only one discrete location along the gage length. Elsewhere, the uncracked concrete shares in resisting flexural tension, resulting in what is known as *tension stiffening*. This tends to reduce curvature. Furthermore, the exact shape of the moment-curvature relation depends strongly upon the reinforcement ratio as well as upon the exact stress-strain curves for the concrete and steel.

Figure 12.13 shows a somewhat simplified moment-curvature diagram for an actual concrete beam section having a tensile reinforcement ratio equal to about one-half the balanced value. The diagram is linear up to the cracking moment M_{cr}, after which a nearly straight line of somewhat flatter slope is obtained. At the moment that initiates yielding M_y, the curvature starts to increase disproportionately. Further increase in applied moment causes extensive inelastic rotation until, eventually, the compressive strain limit of the concrete is reached at the ultimate rotation $\cdot_u$. The maximum moment is often somewhat above the calculated flexural strength M_n, due largely to strain hardening of the reinforcement.

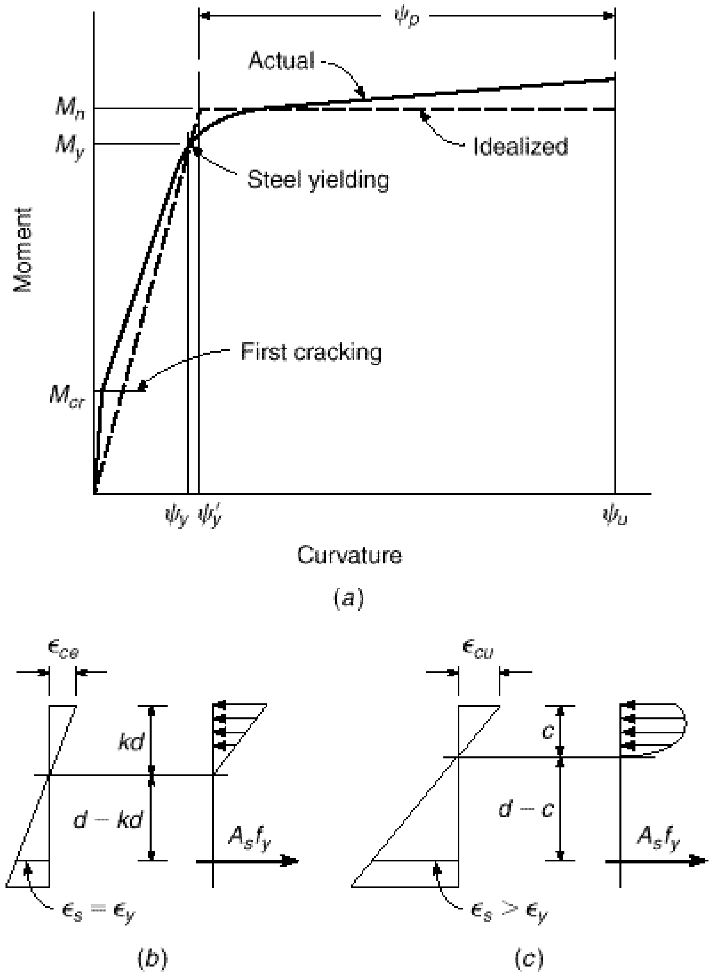


FIGURE 12.13
Plastic hinge characteristics in a reinforced concrete member: (*a*) typical moment-curvature diagram; (*b*) strains and stresses at start of yielding; (*c*) strains and stresses at incipient failure.

The effect of inelastic concrete response prior to steel yielding is small for typically underreinforced sections, as is indicated in Fig. 6.16, and the yield moment can be calculated based on the elastic concrete stress distribution shown in Fig. 12.13*b*:

$$M_y = A_s f_y \left(d - \frac{kd}{3} \right) \tag{12.1}$$

where *kd* is the distance from the compression face to the cracked elastic neutral axis (see Section 3.3b). The nominal moment capacity M_n, based on Fig. 12.13*c*, is calculated by the usual expression

$$M_n = A_s f_y \left(d - \frac{a}{2} \right) = A_s f_y \left(d - \frac{\beta_1 c}{2} \right) \tag{12.2}$$

For purposes of limit analysis, the $M - \psi$ curve is usually idealized, as shown by the dashed line in Fig. 12.13*a*. The slope of the elastic portion of the curve is obtained with satisfactory accuracy using the moment of inertia of the cracked transformed section. After the nominal moment M_n is reached, continued plastic rotation is assumed to occur with no change in applied moment. The elastic curve of the beam will show an abrupt change in slope at such a section. The beam behaves as if there

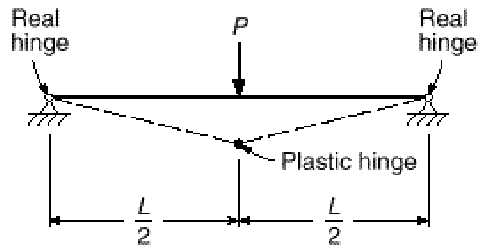


FIGURE 12.14
Statically indeterminate member after the formation of plastic hinge.

were a hinge at that point. However, the hinge will not be "friction-free," but will have a constant resistance to rotation.

If such a plastic hinge forms in a determinate structure, as shown in Fig. 12.14, uncontrolled deflection takes place, and the structure will collapse. The resulting system is referred to as a *mechanism,* an analogy to linkage systems in mechanics. Generalizing, one can say that a statically determinate system requires the formation of only one plastic hinge to become a mechanism.

This is not so for indeterminate structures. In this case, stability may be maintained even though hinges have formed at several cross sections. The formation of such hinges in indeterminate structures permits a redistribution of moments within the beam or frame. It will be assumed for simplicity that the indeterminate beam of Fig. 12.15*a* is symmetrically reinforced, so that the negative bending capacity is the same as the positive. Let the load P be increased gradually until the elastic moment at the fixed support, $\frac{3}{16}PL$, is just equal to the plastic moment capacity of the section M_n. This load is

$$P = P_{el} = \frac{16}{3}\frac{M_n}{L} = 5.33\frac{M_n}{L} \qquad (a)$$

At this load, the positive moment under the load is $\frac{5}{32}PL$, as shown in Fig. 12.15*b*. The beam still responds elastically everywhere but at the left support. At that point the actual fixed support can be replaced for purposes of analysis with a plastic hinge offering a known resisting moment M_n. Because a redundant reaction has been replaced by a known moment, the beam is now determinate.

The load can be increased further until the moment under the load also becomes equal to M_n, at which load the second hinge forms. The structure is converted into a mechanism, as shown in Fig. 12.15*c*, and collapse occurs. The moment diagram at collapse load is shown in Fig. 12.15*d*.

The magnitude of load causing collapse is easily calculated from the geometry of Fig. 12.15*d:*

$$M_n + \frac{M_n}{2} = \frac{PL}{4}$$

from which

$$P = P_n = \frac{6M_n}{L} \qquad (b)$$

By comparison of Eqs. (*b*) and (*a*), it is evident that an increase in P of 12.5 percent is possible, beyond the load that caused the formation of the first plastic hinge, before the beam will actually collapse. Due to the formation of plastic hinges, a redistribution of moments has occurred such that, at failure, the ratio between the positive moment and negative moment is equal to that assumed in reinforcing the structure.

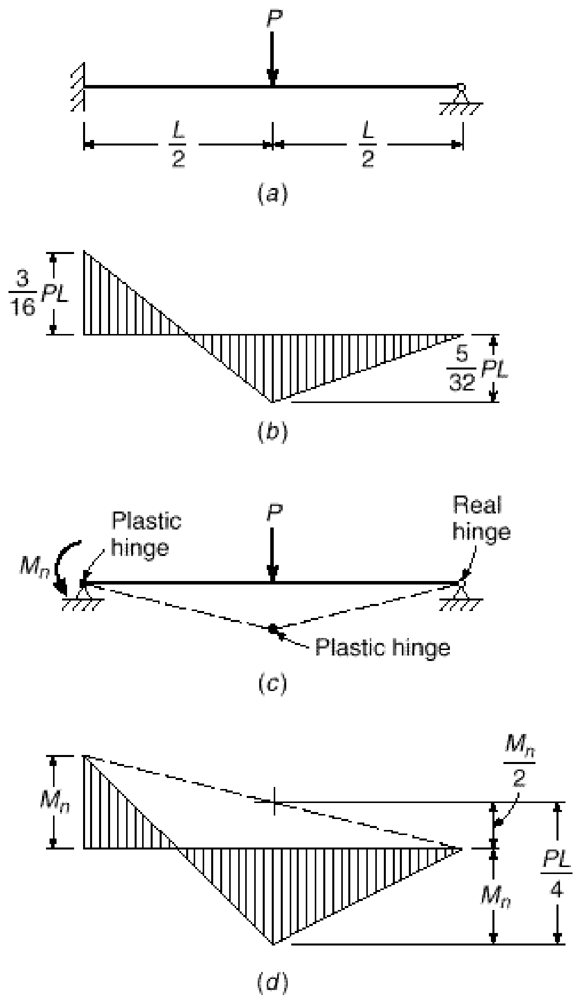


FIGURE 12.15
Indeterminate beam with plastic hinges at support and midspan.

c. Rotation Requirement

It may be evident that there is a direct relation between the amount of redistribution desired and the amount of inelastic rotation at the critical sections of a beam required to produce the desired redistribution. In general, the greater the modification of the elastic-moment ratio, the greater the required rotation capacity to accomplish that change. To illustrate, if the beam of Fig. 12.15*a* had been reinforced according to the elastic-moment diagram of Fig. 12.15*b,* no inelastic-rotation capacity at all would be required. The beam would, at least in theory, yield simultaneously at the left support and at midspan. On the other hand, if the reinforcement at the left support had been deliberately reduced (and the midspan reinforcement correspondingly increased), inelastic rotation at the support would be required before the strength at midspan could be realized.

The amount of rotation required at plastic hinges for any assumed moment diagram can be found by considering the requirements of compatibility. The member must be bent, under the combined effects of elastic moment and plastic hinges, so that the correct boundary conditions are satisfied at the supports. Usually, zero support

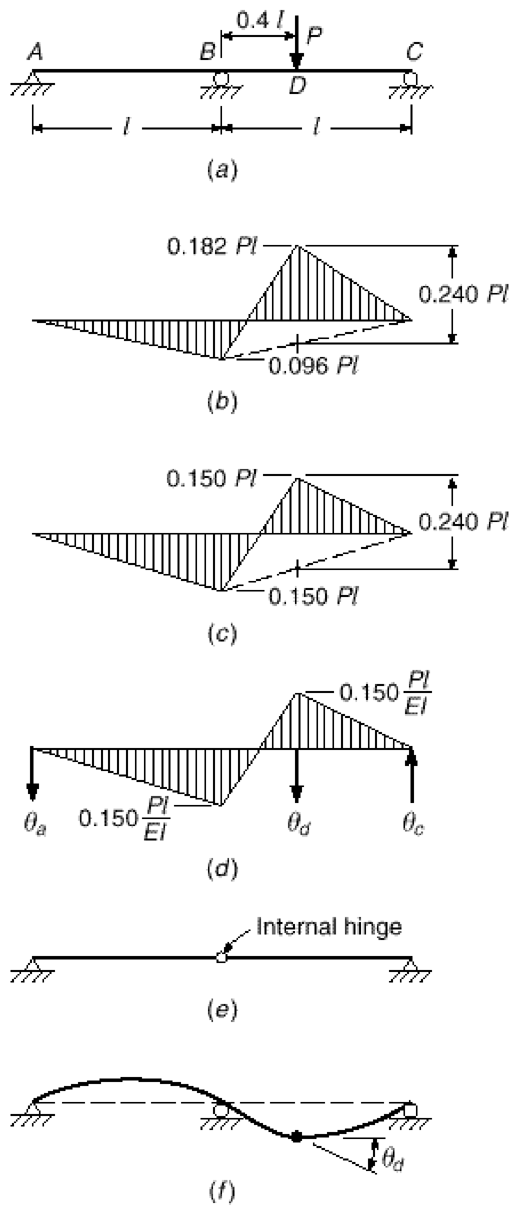


FIGURE 12.16
Moment redistribution in a two-span beam: (*a*) loaded beam; (*b*) elastic moments; (*c*) modified moments; (*d*) *M*·*EI* loads; (*e*) conjugate beam; (*f*) deflection curve.

deflection is to be maintained. Moment-area and conjugate-beam principles are useful in quantitative determination of rotation requirements (Ref. 12.6). In deflection calculations, it is convenient to assume that plastic hinging occurs at a point, rather than being distributed over a finite *hinging length,* as is actually the case. Consequently, in loading the conjugate beam with unit rotations, plastic hinges are represented as concentrated loads.

Calculation of rotation requirements will be illustrated by the two-span continuous beam shown in Fig. 12.16*a*. The elastic-moment diagram resulting from a single concentrated load is shown in Fig. 12.16*b*. The moment at support *B* is 0.096*Pl,* while that under the load is 0.182*Pl*. If the deflection of the beam at support *C* were calculated using the unit rotations equal to M/EI, based on this elastic-moment diagram, a zero result would be obtained.

Figure 12.16*c* shows an alternative, statically admissible moment diagram that was obtained by arbitrarily increasing the support moment from 0.096*Pl* to 0.150*Pl*. If the beam deflection at *C* were calculated using this moment diagram as a basis, a nonzero value would be obtained. This indicates the necessity for inelastic rotation at one or more points to maintain geometric compatibility at the right support.

If the beam were reinforced according to Fig. 12.16*c*, increasing loads would produce the first plastic hinge at *D*, where the beam has been deliberately made understrength. Continued loading would eventually result in formation of the second plastic hinge at *B*, creating a mechanism and leading to collapse of the structure.

Limit analysis requires calculation of rotation at all plastic hinges up to, but not including, the last hinge that triggers actual collapse. Figure 12.16*d* shows the M/EI load to be imposed on the conjugate beam of Fig. 12.16*e*. Also shown is the concentrated angle change θ_d, which is to be evaluated. Starting with the left span, taking moments of the M/EI loads about the internal hinge of the conjugate beam at *B*, one obtains the left reaction of the conjugate beam (equal to the slope of the real beam):

$$\theta_a = 0.025\frac{Pl^2}{EI}$$

With that reaction known, moments are taken about the support *C* of the conjugate beam and set equal to zero to obtain

$$\theta_d = 0.060\frac{Pl^2}{EI}$$

This represents the necessary discontinuity in the slope of the elastic curve shown in Fig. 12.16*f* to restore the beam to zero deflection at the right support. The beam must be capable of developing at least that amount of plastic rotation if the modified moment diagram assumed in Fig. 12.16*c* is to be valid.

d. Rotation Capacity

The capacity of concrete structures to absorb inelastic rotations at plastic-hinge locations is not unlimited. The designer adopting full limit analysis in concrete must calculate not only the amount of rotation required at critical sections to achieve the assumed degree of moment redistribution but also the rotation capacity of the members at those sections to ensure that it is adequate.

Curvature at initiation of yielding is easily calculated from the elastic strain distribution shown in Fig. 12.13*b*.

$$\psi_y = \frac{\epsilon_y}{d(1-k)} \tag{12.3}$$

in which the ratio *k* establishing the depth of the elastic neutral axis is found from Eq. (3.12). The curvature corresponding to the nominal moment can be obtained from the geometry of Fig. 12.13*c*:

$$\psi_u = \frac{\epsilon_{cu}}{c} \tag{12.4}$$

Although it is customary in flexural strength analysis to adopt $\epsilon_{cu} = 0.003$, for purposes of limit analysis a more refined value is needed. Extensive experimental studies

(Refs. 12.7 and 12.8) indicate that the ultimate strain capacity of concrete is strongly influenced by the beam width b, by the moment gradient, and by the presence of additional reinforcement in the form of compression steel and confining steel (i.e., web reinforcement). The last parameter is conveniently introduced by means of a reinforcement ratio ρ'', defined as the ratio of the volume of one stirrup plus its tributary compressive steel volume to the concrete volume tributary to one stirrup. On the basis of empirical studies, the ultimate flexural strain at a plastic hinge is

$$\epsilon_{cu} = 0.003 + 0.02\frac{b}{z} + \left(\frac{\rho'' f_y}{14.5}\right)^2 \tag{12.5}$$

where z is the distance between points of maximum and zero moment. Based on Eqs. (12.3) to (12.5), the inelastic curvature for the idealized relation shown in Fig. 12.13*a* is

$$\psi_p = \psi_u - \psi_y \frac{M_n}{M_y} \tag{12.6}$$

This plastic rotation is not confined to one cross section but is distributed over a finite length referred to as the *hinging length.* The experimental studies upon which Eq. (12.5) is based measured strains and rotations in a length equal to the effective depth d of the test members. Consequently, ϵ_{cu} is an *average* value of ultimate strain over a finite length, and ψ_p, given by Eq. (12.6), is an *average* value of curvature. The total inelastic rotation θ_p can be found by multiplying the average curvature by the hinging length:

$$\theta_p = \left(\psi_u - \psi_y \frac{M_n}{M_y}\right) l_p \tag{12.7}$$

On the basis of current evidence, it appears that the hinging length l_p in support regions, on either side of the support, can be approximated by the expression

$$l_p = 0.5d + 0.05z \tag{12.8}$$

in which z is the distance from the point of maximum moment to the nearest point of zero moment.

e. Moment Redistribution under the ACI Code

Full use of the plastic capacity of reinforced concrete beams and frames requires an extensive analysis of all possible mechanisms and an investigation of rotation requirements and capacities at all proposed hinge locations. The increase in design time may not be justified by the gains obtained. On the other hand, a restricted amount of redistribution of elastic moments can safely be made without complete analysis, yet may be sufficient to obtain most of the advantages of limit analysis.

A limited amount of redistribution is permitted by ACI Code 8.4, depending upon a rough measure of available ductility, without explicit calculation of rotation requirements and capacities. The net tensile strain in the extreme tension steel at nominal strength ϵ_t, given in Eq. (3.29), is used as an indicator of rotation capacity. Accordingly, ACI Code 8.4 provides as follows:

> Except where approximate values for moments are used, the negative moments calculated by elastic theory at the supports of continuous flexural members for any assumed loading arrangement may be increased or decreased by not more than $1000\epsilon_t$ percent, with

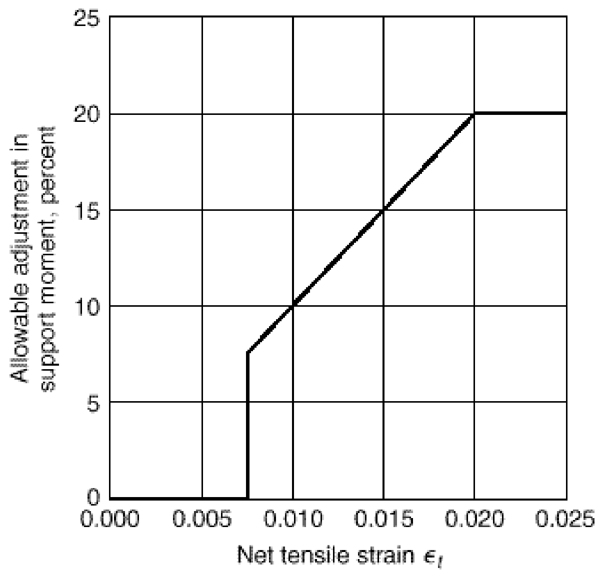


FIGURE 12.17
Allowable moment redistribution under the ACI Code.

a maximum of 20 percent. These modified negative moments shall be used for calculation of the moments at sections within the spans. Such an adjustment shall be made only when ·$_t$ is equal to or greater than 0.0075 at the section at which the moment is reduced.

Redistribution for values of ·$_t$ < 0.0075 is conservatively prohibited. The ACI Code provisions are shown graphically in Fig. 12.17. The value of · corresponding to a given value of ·$_t$, and thus a given percentage change in moment, can be calculated using Eq. (3.30*a*) from Section 3.4d.

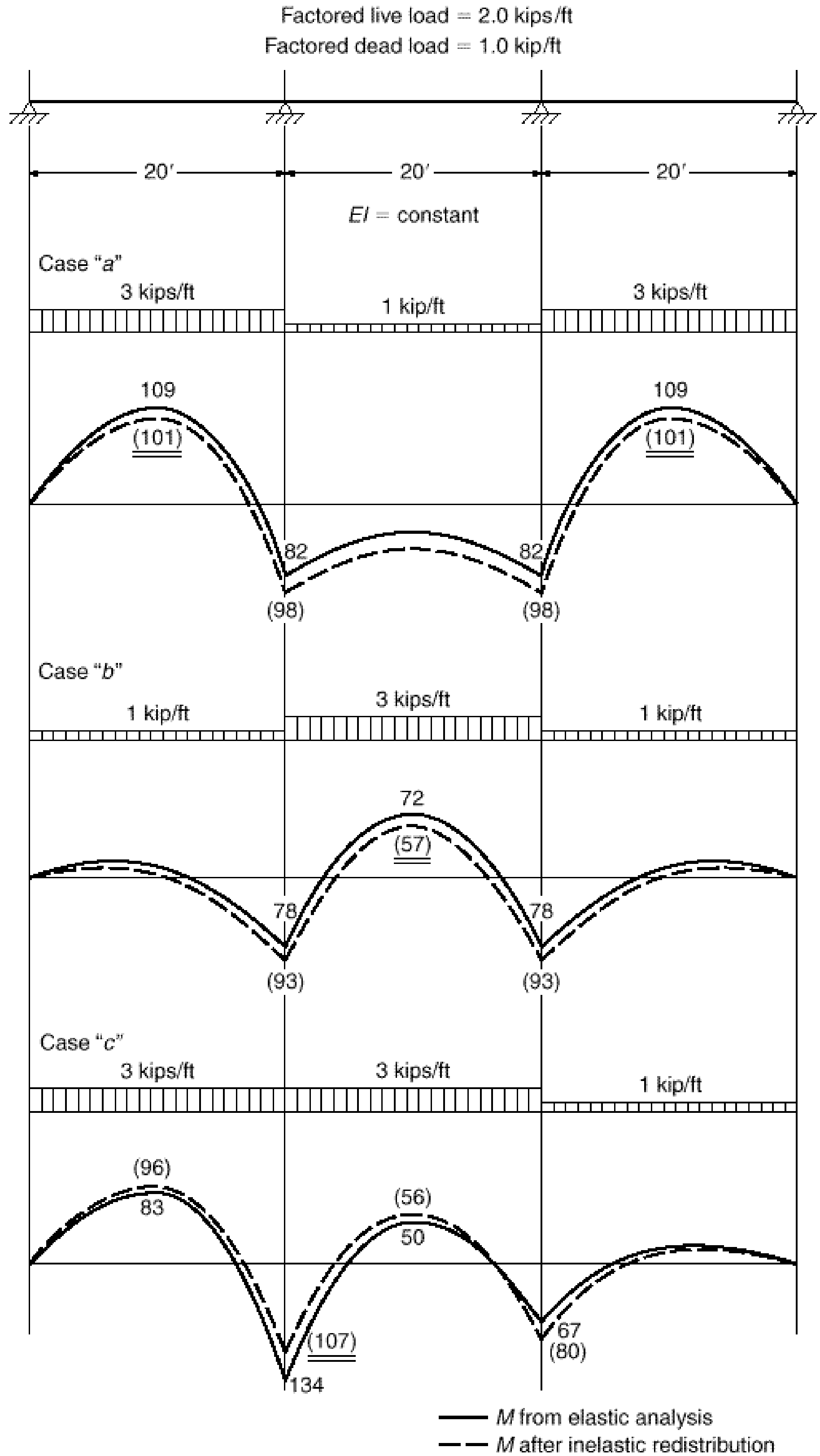
To demonstrate the advantage of moment redistribution when alternative loadings are involved, consider the concrete beam of Fig. 12.18. A three-span continuous beam is shown, with dead load of 1 kip/ft and live load of 2 kips/ft. To obtain maximum moments at all critical design sections, it is necessary to consider three alternative loadings. Case *a,* with live and dead load over exterior spans and dead load only over the interior span, will produce the maximum positive moment in the exterior spans. Case *b,* with dead load on exterior spans and dead and live load on the interior span, will produce the maximum positive moment in the interior span. The maximum negative moment over the interior support is obtained by placing dead and live load on the two adjacent spans and dead load only on the far exterior span, as shown in case *c.*

It will be assumed for simplicity that a 20 percent adjustment of support moments is permitted throughout, provided span moments are modified accordingly. An overall reduction in design moments through the entire three-span beam may be possible. Case *a,* for example, produces an elastic maximum span moment in the exterior spans of 109 ft-kips. Corresponding to this is an elastic negative moment of 82 ft-kips at the interior support. Adjusting the support moment upward by 20 percent, one obtains a negative moment of 98 ft-kips, which results in a downward adjustment of the span moment to 101 ft-kips.

Now consider case *b.* By a similar redistribution of moments, a reduced middle-span moment of 57 ft-kips is obtained through an increase of the support moment from 78 to 93 ft-kips.

The moment obtained at the first interior support for loading case *c* can be adjusted in the reverse direction; i.e., the support moment is decreased by 20 percent

FIGURE 12.18
Redistribution of moments in a three-span continuous beam.



to 107 ft-kips. To avoid increasing the controlling span moment of the interior span, the right interior support moment is adjusted upward by 20 percent to 80 ft-kips. The positive moments in the left exterior span and in the interior span corresponding to these modified support moments are 96 and 56 ft-kips, respectively.

It will be observed that the reduction obtained for the span moments in cases *a* and *b* were achieved at the expense of increasing the moment at the first interior support. However, the increased support moment in each case was less than the moment for which that support would have to be designed based on the loading *c*, which produced the maximum support moment. Similarly, the reduction in support moment in case *c* was taken at the expense of an increase in span moments in the two adjacent spans. However, in each case the increased span moments were less than the maximum span moments obtained for other loading conditions. The final design moments at all critical sections are underlined in Fig. 12.18. It can be seen, then, that the net result is a reduction in design moments over the entire beam. This modification of moments does not mean a reduction in safety factor below that implied in code safety provisions; rather, it means a reduction of the *excess* strength that would otherwise be present in the structure because of the actual redistribution of moments that would occur before failure. It reflects the fact that the maximum design moments are obtained from alternative load patterns, which could not exist concurrently. The end result is a more realistic appraisal of the actual collapse load of the indeterminate structure.

12.10 Conclusion

The problems associated with analysis of reinforced concrete structures are many. The engineer must not only accept the uncertainties of load placement, magnitude, and duration typical of any structural analysis, but must also cope with other complications that are unique to reinforced concrete. These are mainly associated with estimation of moment of inertia of the reinforced concrete sections and with the influence of concrete creep. They may be summarized briefly as follows: (1) effective moments of inertia change depending on the sign of the bending moment, (2) moments of inertia depend not only on the effective concrete section, but also on the steel, a part of which may be discontinuous, (3) moments of inertia depend on cracking, which is both location-dependent and load-dependent, and (4) the concrete is subject to creep under sustained loads, reducing its effective modulus. In addition, joint restraints and conditions of support for complex structures are seldom completely in accordance with the idealization. The student may well despair of accurate calculation of the internal forces for which the members of a reinforced concrete frame must be designed.

It may be reassuring to know that reinforced concrete has a remarkable capacity to adapt to the assumptions of the designer. This has been pointed out by a number of outstanding engineers. *Luigi Nervi*, the renowned Italian architect-engineer, has stated it eloquently as follows:

> Mainly because of plastic flow, a concrete structure tries with admirable docility to adapt itself to our calculations—which do not always represent the most logical and spontaneous answer to the request of the forces at play—and even tries to correct our deficiencies and errors. Sections and regions too highly stressed yield and channel some of their loads to other sections or regions, which accept this additional task with commendable spirit of collaboration, within the limits of their own strength.[†]

[†] P. L. Nervi, *Structures*, F. W. Dodge Corp., New York, 1956.

Hardy Cross, best known for his development of the moment distribution method of analysis (see Section 12.4), noted the beneficial effects of concrete creep, by which a structure can adapt to support settlements, which, on the basis of elastic analysis, cause forces and movements sufficient to fail the structure. *Halvard Birkeland,* one of the pioneers in the development of prestressed concrete in the United States, referred to the "wisdom of the structure," noting that ". . . the structure, in many instances, will accept our rash assumptions and our imperfect mathematical models . . . the structure will exhaust all means of standing before it decides to fall."†

Thus it may be of some comfort to know that *a reinforced concrete structure will tend to act as the engineer has assumed it will act.* Reasonable assumptions in the analysis may safely be made. But corollary to this important principle is the acceptance of its limits: *the general pattern of forces and moments must be recognized, and at least one reasonable load path provided.* Too great a deviation from the actual distribution of internal forces can result in serviceability problems associated with cracking and deflection, and can even result in premature failure. It is for this reason that methods of limit analysis for reinforced concrete include restrictions on the amount of redistribution of elastic moments (see Section 12.9). But it is reassuring to know that, if good judgment is used in assigning internal forces to critical sections, the *wisdom of the structure* will prevail.

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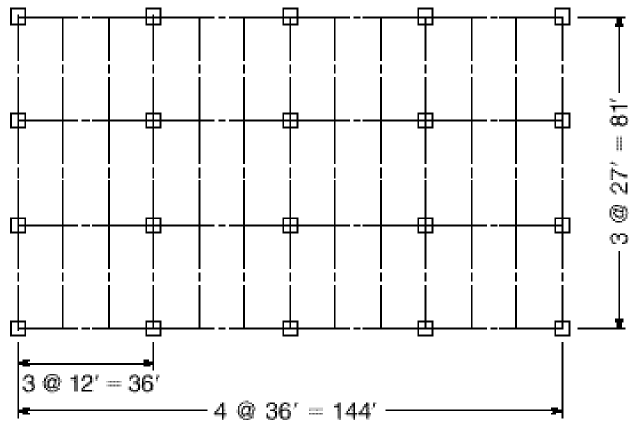
Problems

12.1. Complete the preliminary design of the four-story heavy storage facility shown in Fig. P12.1. The floor live load is 250 psf, the roof live load is 12 psf, and the dead load on all floors and the roof consists of the structure self-weight plus 10 psf for utilities. The building is enclosed in a self-supporting curtain wall that also carries the lateral load on the structure. Beams are spaced at 12 ft; girders are spaced at 27 ft. The minimum clear space between floors is 11 ft, and the floor depth should not exceed 30 in. The column cross sections should be maintained from floor to floor. Use $f_y = 60{,}000$ psi and $f'_c = 4000$ psi for the floors. Concrete with f'_c up to 8000 psi is available for the columns. The preliminary design should include the initial dimensions of the structural slab, beams, girders, and columns for a typical floor.

† H. L. Birkeland, "The Wisdom of the Structure," *J. ACI,* April 1978, pp. 105–111.

FIGURE P12.1



12.2. A concrete beam with $b = 12$ in., $h = 26.5$ in., and $d = 24$ in., having a span of 24 ft, can be considered fully fixed at the left support and supported vertically but with no rotational restraint (e.g., roller) at the right end. It is reinforced for positive bending with a combination of bars giving $A_s = 2.45$ in^2, and for negative bending at the left support with $A_s = 2.88$ in^2. Positive bars are carried 6 in. into the face of the left support, according to the ACI Code requirements, but lack the embedded length to be considered effective as compression steel. No. 3 (No. 10) closed hoop stirrups are provided at 9 in. spacing over the full span. The factored load consists of a single concentrated force of 63.3 kips at midspan. Self-weight of the beam may be neglected in the calculations. Calculate the rotation requirement at the first plastic hinge to form (*a*) if the beam is reinforced according to the description above, (*b*) if, to reduce bar congestion at the left support, that steel area is reduced by 12.5 percent, with an appropriate increase in the positive steel area, and (*c*) if the steel area at the left support is reduced by 25 percent, compared with the original description, with an appropriate increase in the positive steel area. Also calculate the rotation capacity of the critical section, for comparison with the requirements of (*a*), (*b*), and (*c*). Comment on your results and compare with the approach to moment redistribution presented in the ACI Code. Material strengths are $f_y = 60$ ksi and $f'_c = 4$ ksi.

12.3. A 12-span continuous reinforced concrete T beam is to carry a calculated dead load of 900 lb/ft including self-weight, plus a service live load of 1400 lb/ft on uniform spans measuring 26.5 ft between centers of supporting columns (25 ft clear spans). The slab thickness is 6 in., and the effective flange width is 75 in. Web proportions are $b_w = 0.6d$, and the maximum reinforcement ratio will be set at 0.011. All columns will be 18 in. square. Material strengths are $f'_c =$ 4000 psi and $f_y = 60{,}000$ psi.

(*a*) Find the factored moments for the exterior and first interior span based on the ACI Code moment coefficients of Table 12.1.

(*b*) Find the factored moments in the exterior and first interior span by elastic frame analysis, assuming the floor-to-floor height to be 10 ft. Note that alternative live loadings should be considered (see Section 12.2*a*) and that moments can be reduced to account for the support width (see Section 12.5*a*). Compare your results with those obtained using the ACI moment coefficients.

(*c*) Adjust the factored negative and positive moments, taking advantage of the redistribution provisions of the ACI Code. Assume that a 10 percent minimum redistribution is possible.

(*d*) Design the exterior and first interior spans for flexure and shear, finding concrete dimensions and bar requirements, basing your design on the assumptions and modified moments in part (*c*).

12.4. A continuous reinforced concrete frame consists of a two-span rectangular beam *ABC*, with center-to-center spans *AB* and *BC* of 24 ft. Columns measuring 14 in. square are provided at *A*, *B*, and *C*. The columns may be considered fully fixed at the floors above and below for purposes of analysis. The beam will carry a service live load of 1200 lb/ft and a calculated dead load of 1000 lb/ft, including self-weight. Floor-to-floor height is 12 ft. Material strengths are $f_y = 60{,}000$ psi and $f_c' = 4000$ psi.

(*a*) Carry out an elastic analysis of the two-span frame, considering alternate live loadings to maximize the bending moment at all critical sections. Design the beams, using a maximum reinforcement ratio of 0.012 and $d = 2b$. Find the required concrete section and required steel areas at positive and negative bending sections. Select the reinforcement. Cutoff points can be determined according to Fig. 5.15*a*. Note that negative design moments are at the face of supports, not support centerlines.

(*b*) Take maximum advantage of the redistribution provisions of ACI Code 8.4 (see Section 12.9*e*) to reduce design moments at all critical sections, and redesign the steel for the beams. Keep the concrete section unchanged. Select reinforcement and determine cutoff points.

(*c*) Comment on your two designs with regard to the amount of steel required and the possible congestion of steel at the critical bending sections. You may assume that the shear reinforcement is unchanged in the redesigned beam.