14 YIELD LINE ANALYSIS FOR SLABS

14.1 INTRODUCTION

Most concrete slabs are designed for moments found by the methods described in Chapter 13. These methods are based essentially upon elastic theory. On the other hand, reinforcement for slabs is calculated by strength methods that account for the actual inelastic behavior of members at the factored load stage. A corresponding contradiction exists in the process by which beams and frames are analyzed and designed, as was discussed in Section 12.9, and the concept of limit, or plastic, analysis of reinforced concrete was introduced. Limit analysis not only eliminates the inconsistency of combining elastic analysis with inelastic design but also accounts for the reserve strength characteristic of most reinforced concrete structures and permits, within limits, an arbitrary readjustment of moments found by elastic analysis to arrive at design moments that permit more practical reinforcing arrangements.

For slabs, there is still another good reason for interest in limit analysis. The elasticity-based methods of Chapter 13 are restricted in important ways. Slab panels must be square or rectangular. They must be supported along two opposite sides (one-way slabs), two pairs of opposite sides (two-way edge-supported slabs) or by a fairly regular array of columns (flat plates and related forms). Loads must be uniformly distributed, at least within the bounds of any single panel. There can be no large openings. But in practice, many slabs do not meet these restrictions. Answers are needed, for example, for round or triangular slabs, slabs with large openings, slabs supported on two or three edges only, and slabs carrying concentrated loads. Limit analysis provides a powerful and versatile tool for treating such problems.

It was evident from the discussion of Section 12.9 that full plastic analysis of a continuous reinforced concrete beam or frame would be tedious and time consuming because of the need to calculate the rotation requirement at all plastic hinges and to check rotation capacity at each hinge to ensure that it is adequate. Consequently, for beams and frames, the very simplified approach to plastic moment redistribution of ACI Code 8.4 is used. However, for slabs, which typically have tensile reinforcement ratios much below the balanced value and consequently have large rotation capacity, it can be safely assumed that the necessary ductility is present. Practical methods for the plastic analysis of slabs are thus possible and have been developed. *Yield line theory*, presented in this chapter, is one of these. Although the ACI Code contains no specific provisions for limit or plastic analysis of slabs, ACI Code 1.4 permits use of "any system of design or construction," the adequacy of which has been shown by successful use, analysis, or tests, and ACI Code Commentary 13.5.1 refers specifically to yield line analysis as an acceptable approach.

Yield line analysis for slabs was first proposed by Ingerslev (Ref. 14.1) and was greatly extended by Johansen (Refs. 14.2 and 14.3). Early publications were mainly in Danish, and it was not until Hognestad's English language summary (Ref. 14.4) of Johansen's work that the method received wide attention. Since that time, a number of important publications on the method have appeared (Refs. 14.5 through 14.15). A particularly useful and comprehensive treatment will be found in Ref. 14.15.

The *plastic hinge* was introduced in Section 12.9 as a location along a member in a continuous beam or frame at which, upon overloading, there would be large inelastic rotation at essentially a constant resisting moment. For slabs, the corresponding mechanism is the *yield line*. For the overloaded slab, the resisting moment per unit length measured along a yield line is constant as inelastic rotation occurs; the yield line serves as an axis of rotation for the slab segment.

Figure 14.1*a* shows a simply supported, uniformly loaded reinforced concrete slab. It will be assumed to be underreinforced (as are almost all slabs), with · < ·$_b$. The elastic moment diagram is shown in Fig. 14.1*b*. As the load is increased, when the applied moment becomes equal to the flexural capacity of the slab cross section, the tensile steel starts to yield along the transverse line of maximum moment.

Upon yielding, the curvature of the slab at the yielding section increases sharply, and deflection increases disproportionately. The elastic curvatures along the slab span are small compared with the curvature resulting from plastic deformation at the yield line, and it is acceptable to consider that the slab segments between the yield line and supports remain rigid, with all the curvature occurring at the yield line, as shown in Fig. 14.1*c*. The "hinge" that forms at the yield line rotates with essentially constant resistance, according to the relation shown earlier in Fig. 12.13*a*. The resistance per unit width of slab is the nominal flexural strength of the slab; that is, $m_p = m_n$, where

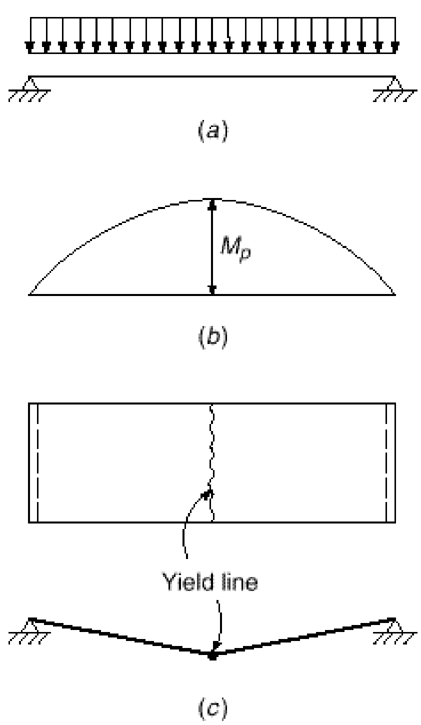
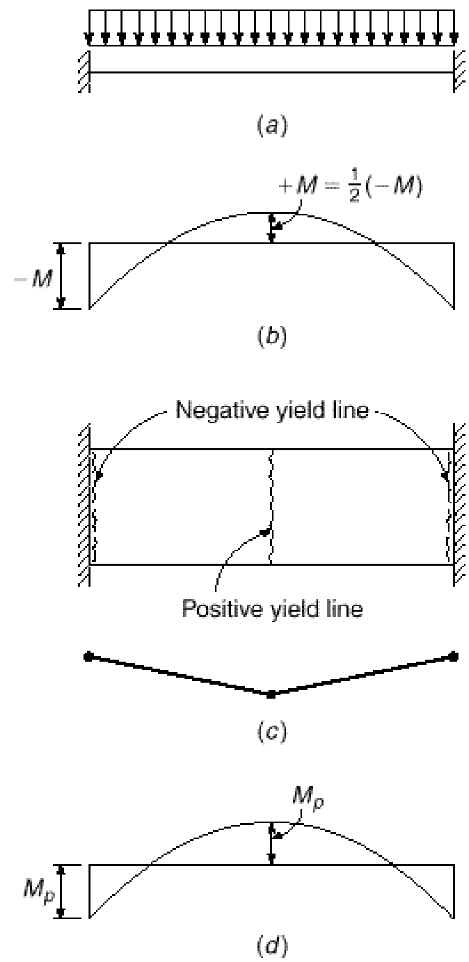


FIGURE 14.1
Simply supported, uniformly loaded one-way slab.

FIGURE 14.2
Fixed-end, uniformly loaded one-way slab.



m_n is calculated by the usual equations. For design purposes, m_p would be taken equal to ϕm_n, with ϕ typically equal to 0.90, since ρ is well below ρ_{max} for most slabs.

For a statically determinate slab like that in Fig. 14.1, the formation of one yield line results in collapse. A "mechanism" forms, i.e., the segments of the slab between the hinge and the supports are able to move without an increase in load. Indeterminate structures, however, can usually sustain their loads without collapse even after the formation of one or more yield lines. When it is loaded uniformly, the fixed-fixed slab in Fig. 14.2*a*, assumed here to be equally reinforced for positive and negative moments, will have an elastic distribution of moments, as shown in Fig. 14.2*b*. As the load is gradually increased, the more highly stressed sections at the support start yielding. Rotations occur at the support line hinges, but restraining moments of constant value m_p continue to act. The load can be increased further, until the moment at midspan becomes equal to the moment capacity there, and a third yield line forms, as shown in Fig. 14.2*c*. The slab is now a mechanism, large deflections occur, and collapse takes place.

The moment diagram just before failure is shown in Fig. 14.2*d*. Note that the ratio of elastic positive to negative moments of 1:2 no longer holds. Due to inelastic deformation, the ratio of these moments just before collapse is 1:1 for this particular structure. Redistribution of moments was discussed earlier in Section 12.9, and it was pointed out that the moment ratios at the collapse stage depend upon the reinforcement provided, not upon the results of elastic analysis.

14.2 Upper and Lower Bound Theorems

Plastic analysis methods such as the yield line theory derive from the general theory of structural plasticity, which states that the collapse load of a structure lies between two limits, an upper bound and a lower bound of the true collapse load. These limits can be found by well-established methods. A full solution by the theory of plasticity would attempt to make the lower and upper bounds converge to a single correct solution.

The lower bound theorem and the upper bound theorem, when applied to slabs, can be stated as follows:

> **Lower bound theorem:** If, for a given external load, it is possible to find a distribution of moments that satisfies equilibrium requirements, with the moment not exceeding the yield moment at any location, and if the boundary conditions are satisfied, then the given load is a lower bound of the true carrying capacity.

> **Upper bound theorem:** If, for a small increment of displacement, the internal work done by the slab, assuming that the moment at every plastic hinge is equal to the yield moment and that boundary conditions are satisfied, is equal to the external work done by the given load for that same small increment of displacement, then that load is an upper bound of the true carrying capacity.

If the lower bound conditions are satisfied, the slab can certainly carry the given load, although a higher load may be carried if internal redistributions of moment occur. If the upper bound conditions are satisfied, a load greater than the given load will certainly cause failure, although a lower load may produce collapse if the selected failure mechanism is incorrect in any sense.

In practice, in the plastic analysis of structures, one works either with the lower bound theorem or the upper bound theorem, not both, and precautions are taken to ensure that the predicted failure load at least closely approaches the correct value.

The yield line method of analysis for slabs is an upper bound method, and consequently, the failure load calculated for a slab with known flexural resistances may be higher than the true value. This is certainly a concern, as the designer would naturally prefer to be correct, or at least on the safe side. However, procedures can be incorporated in yield line analysis to help ensure that the calculated capacity is correct. Such procedures will be illustrated by the examples in Sections 14.4 and 14.5.

14.3 Rules for Yield Lines

The location and orientation of the yield line were evident for the simple slab in Fig. 14.1. Similarly, the yield lines were easily established for the one-way indeterminate slab in Fig. 14.2. For other cases, it is helpful to have a set of guidelines for drawing yield lines and locating axes of rotation. When a slab is on the verge of collapse because of the existence of a sufficient number of real or plastic hinges to form a mechanism, axes of rotation will be located along the lines of support or over point supports such as columns. The slab segments can be considered to rotate as rigid bodies in space about these axes of rotation. The yield line between any two adjacent slab segments is a straight line, being the intersection of two essentially plane surfaces. Because the yield line (as a line of intersection of two planes) contains all points common to these two planes, it must contain the point of intersection (if any) of the two axes of rotation, which is also common to the two planes. That is, the yield line (or yield line extended) must pass through the point of intersection of the axes of rotation of the two adjacent slab segments.

The terms *positive yield line* and *negative yield line* are used to distinguish between those associated with tension at the bottom and tension at the top of the slab, respectively.

Guidelines for establishing axes of rotation and yield lines are summarized as follows:

1. Yield lines are straight lines because they represent the intersection of two planes.
2. Yield lines represent axes of rotation.
3. The supported edges of the slab will also establish axes of rotation. If the edge is fixed, a negative yield line may form providing constant resistance to rotation. If the edge is simply supported, the axis of rotation provides zero restraint.
4. An axis of rotation will pass over any column support. Its orientation depends on other considerations.
5. Yield lines form under concentrated loads, radiating outward from the point of application.
6. A yield line between two slab segments must pass through the point of intersection of the axes of rotation of the adjacent slab segments.

In Fig. 14.3, which shows a slab simply supported along its four sides, rotation of slab segments *A* and *B* is about *ab* and *cd*, respectively. The yield line *ef* between these two segments is a straight line passing through *f*, the point of intersection of the axes of rotation.

Illustrations are given in Fig. 14.4 of the application of the guidelines to the establishment of yield line locations and failure mechanisms for a number of slabs with various support condition. Figure 14.4*a* shows a slab continuous over parallel supports. Axes of rotation are situated along the supports (negative yield lines) and near midspan, parallel to the supports (positive yield line). The particular location of the positive yield line in this case and the other cases in Fig. 14.4 depends upon the distribution of loading and the reinforcement of the slab. Methods for determining its location will be discussed later.

For the continuous slab on nonparallel supports, shown in Fig. 14.4*b*, the midspan yield line (extended) must pass through the intersection of the axes of rotation over the supports. In Fig. 14.4*c* there are axes of rotation over all four simple supports.

FIGURE 14.3
Two-way slab with simply supported edges.

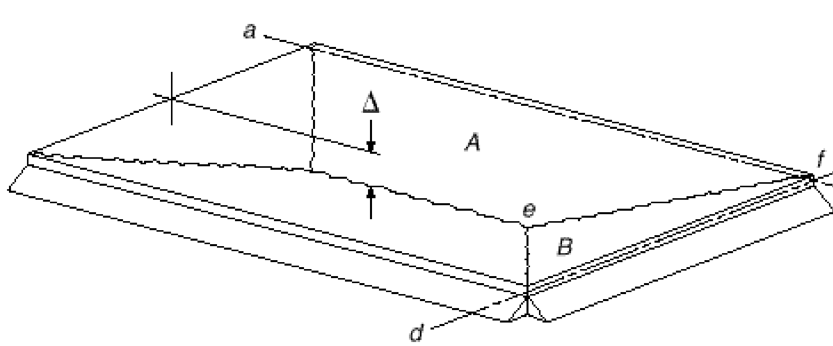
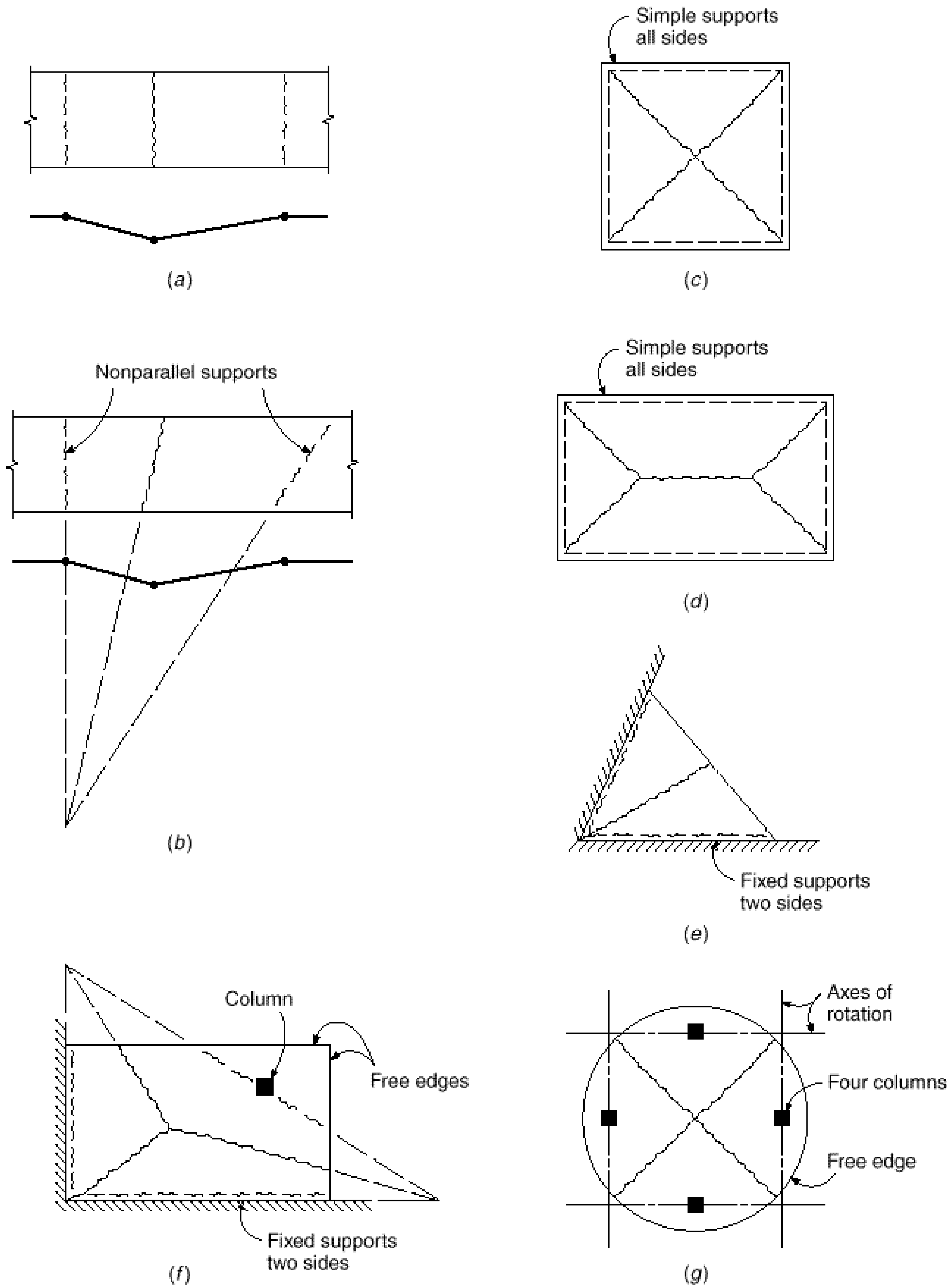


FIGURE 14.4
Typical yield line patterns.



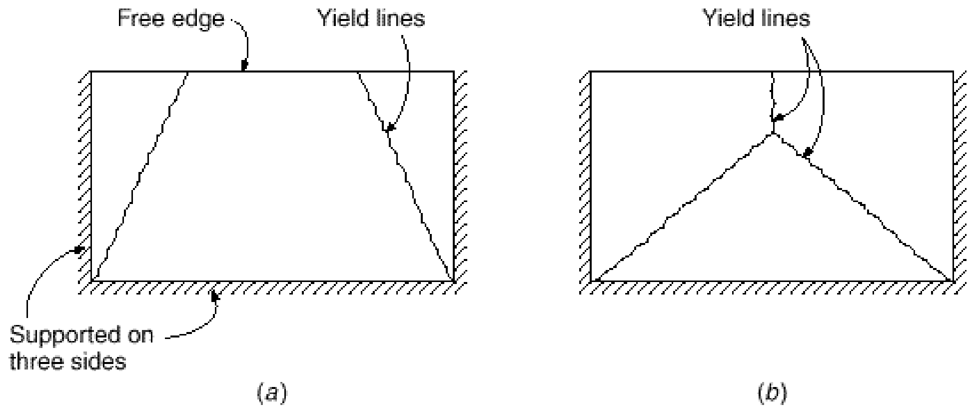


FIGURE 14.5
Alternative mechanisms for a slab supported on three sides.

Positive yield lines form along the lines of intersection of the rotating segments of the slab. A rectangular two-way slab on simple supports is shown in Fig. 14.4*d*. The diagonal yield lines must pass through the corners, while the central yield line is parallel to the two long sides (axes of rotation along opposite supports intersect at infinity in this case).

With this background, the reader should have no difficulty in applying the guidelines to the slabs in Fig. 14.4*e* to *g* to confirm the general pattern of yield lines shown. Many other examples will be found in Refs. 14.1 to 14.15.

Once the general pattern of yielding and rotation has been established by applying the guidelines just stated, the specific location and orientation of the axes of rotation and the failure load for the slab can be established by either of two methods. The first will be referred to as the *method of segment equilibrium* and will be presented in Section 14.4. It requires consideration of the equilibrium of the individual slab segments forming the collapse mechanism and leads to a set of simultaneous equations permitting solution for the unknown geometric parameters and for the relation between load capacity and resisting moments. The second, the *method of virtual work*, will be described in Section 14.5. This method is based on equating the internal work done at the plastic hinges with the external work done by the loads as the predefined failure mechanism is given a small virtual displacement.

It should be emphasized that *either method of yield line analysis is an upper bound approach* in the sense that the true collapse load will never be higher, but may be lower, than the load predicted. For either method, the solution has two essential parts: (*a*) establishing the correct failure pattern, and (*b*) finding the geometric parameters that define the exact location and orientation of the yield lines and solving for the relation between applied load and resisting moments. Either method can be developed in such a way as to lead to the correct solution for the mechanism chosen for study, but the true failure load will be found only if the correct mechanism has been selected.

For example, the rectangular slab in Fig. 14.5, supported along only three sides and free along the fourth, may fail by either of the two mechanisms shown. An analysis based on yield pattern *a* may indicate a slab capacity higher than one based on pattern *b*, or vice versa. It is necessary to investigate *all possible mechanisms* for any slab to confirm that the correct solution, giving the lowest failure load, has been found.[†]

[†] The importance of this point was underscored by Professor Arne Hillerborg, of Lund Institute of Technology, Sweden, in a letter to the editor of the ACI publication *Concr. Intl*, vol. 13, no. 5, 1991. Professor Hillerborg noted that, in reality, there are two additional yield line patterns for a slab such as shown in Fig. 14.5. For a particular set of dimensions and reinforcement, both of these gave a lower failure load than did the mechanism shown in Fig. 14.5*a*.

The method of segment equilibrium should not be confused with a true equilibrium method such as the strip method described in Chapter 15. A true equilibrium method is a lower bound method of analysis—i.e., it will always give a *lower bound* of the true capacity of the slab.

14.4 Analysis by Segment Equilibrium

Once the general pattern of yielding and rotation has been established by applying the guidelines of Section 14.3, the location and orientation of axes of rotation and the failure load for the slab can be established based on the equilibrium of the various segments of the slab. Each segment, studied as a free body, must be in equilibrium under the action of the applied loads, the moments along the yield lines, and the reactions or shear along the support lines. Because the yield moments are principal moments, twisting moments are zero along the yield lines, and in most cases the shearing forces are also zero. Only the unit moment m generally is considered in writing equilibrium equations.

EXAMPLE 14.1 **Segment equilibrium analysis of one-way slab.** The method will be demonstrated first with respect to the one-way, uniformly loaded, continuous slab of Fig. 14.6a. The slab has a 10 ft span and is reinforced to provide a resistance to positive bending · m_n = 5.0 ft-kips/ft through the span. In addition, negative steel over the supports provides moment capacities of 5.0 ft-kips/ft at A and 7.5 ft-kips/ft at C. Determine the load capacity of the slab.

Solution. The number of equilibrium equations required will depend upon the number of unknowns. One unknown is always the relation between the resisting moments of the slab and the load. Other unknowns are needed to define the locations of yield lines. In the present instance, one additional equation will suffice to define the distance of the yield line from

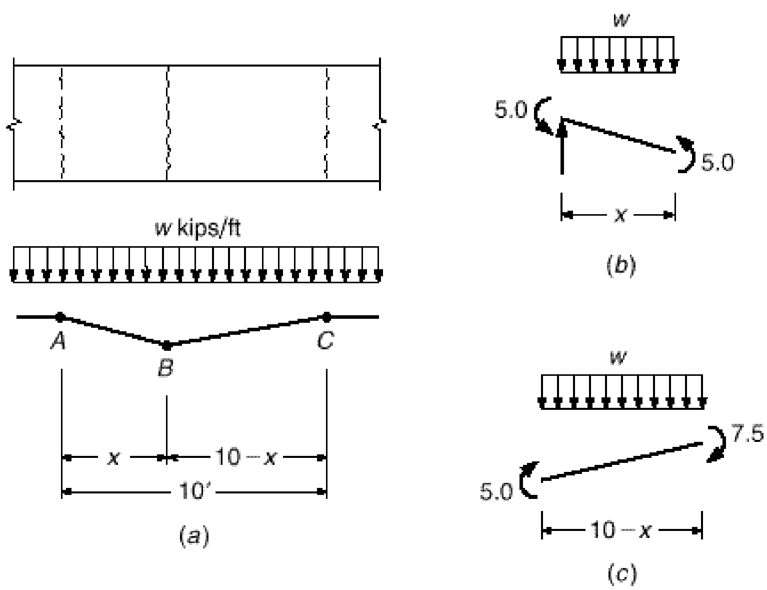


FIGURE 14.6
Analysis of a one-way slab by segment equilibrium equations.

the supports. Taking the left segment of the slab as a free body and writing the equation for moment equilibrium about the left support line (see Fig. 14.6*b*) leads to

$$\frac{wx^2}{2} - 10.0 = 0 \qquad (a)$$

Similarly, for the right slab segment,

$$\frac{w}{2}(10 - x)^2 - 12.5 = 0 \qquad (b)$$

Solving Eqs. (*a*) and (*b*) simultaneously for w and x results in

$$w = 0.89 \text{ kips/ft}^2 \qquad x = 4.75 \text{ ft}$$

If a slab is reinforced in orthogonal directions so that the resisting moment is the same in these two directions, the moment capacity of the slab will be the same along any other line, regardless of direction. Such a slab is said to be *isotropically* reinforced. If, however, the strengths are different in two perpendicular directions, the slab is called *orthogonally anisotropic*, or simply *orthotropic*. Only isotropic slabs will be discussed in this section. Orthotropic reinforcement, which is very common in practice, will be discussed in Section 14.6.

It is convenient in yield line analysis to represent moments with vectors. The standard convention, in which the moment acts in a clockwise direction when viewed along the vector arrow, will be followed. Treatment of moments as vector quantities will be illustrated by the following example:

EXAMPLE 14.2

Segment equilibrium analysis of square slab. A square slab is simply supported along all sides and is to be isotropically reinforced. Determine the resisting moment $m = \phi m_n$ per linear foot required just to sustain a uniformly distributed factored load of w psf.

Solution. Conditions of symmetry indicate the yield line pattern shown in Fig. 14.7*a*. Considering the moment equilibrium of any one of the identical slab segments about its support (see Fig. 14.7*b*), one obtains

$$\frac{wL^2}{4}\frac{L}{6} - 2\left(\frac{mL}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 0$$

$$m = \frac{wL^2}{24}$$

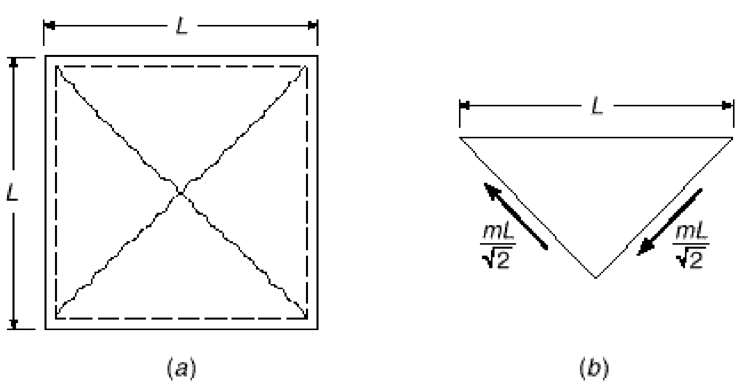


FIGURE 14.7
Analysis of a square two-way slab by segment equilibrium equations.

In both examples just given, the resisting moment was constant along any particular yield line, i.e., the reinforcing bars were of constant diameter and equally spaced along a given yield line. On the other hand, it will be recalled that, by the elastic methods of slab analysis presented in Chapter 13, reinforcing bars generally have a different spacing and may be of different diameter in middle strips compared with column or edge strips. A slab designed by elastic methods, leading to such variations, can easily be analyzed for strength by the yield line method. It is merely necessary to subdivide a yield line into its component parts, within any one of which the resisting moment per unit length of hinge is constant. Either the equilibrium equations of this section or the work equations of Section 14.5 can be modified in this way.

14.5 Analysis by Virtual Work

Alternative to the method of Section 14.4 is a method of analysis using the principle of virtual work. Since the moments and loads are in equilibrium when the yield line pattern has formed, an infinitesimal increase in load will cause the structure to deflect further. The external work done by the loads to cause a small arbitrary virtual deflection must equal the internal work done as the slab rotates at the yield lines to accommodate this deflection. The slab is therefore given a virtual displacement, and the corresponding rotations at the various yield lines can be calculated. By equating internal and external work, the relation between the applied loads and the resisting moments of the slab is obtained. Elastic rotations and deflections are not considered when writing the work equations, as they are very small compared with the plastic deformations.

a. External Work Done by Loads

An external load acting on a slab segment, as a small virtual displacement is imposed, does work equal to the product of its constant magnitude and the distance through which the point of application of the load moves. If the load is distributed over a length or an area, rather than concentrated, the work can be calculated as the product of the total load and the displacement of the point of application of its resultant.

Figure 14.8 illustrates the basis for external work calculation for several types of loads. If a square slab carrying a single concentrated load at its center (Fig. 14.8*a*) is given a virtual displacement defined by a unit value under the load, the external work is

$$W_e = P \times 1 \tag{a}$$

If the slab shown in Fig. 14.8*b*, supported along three sides and free along the fourth, is loaded with a line load w per unit length along the free edge, and if that edge is given a virtual displacement having unit value along the central part, the external work is

$$W_e = (2wa) \times \frac{1}{2} + wb = w(a + b) \tag{b}$$

When a distributed load w per unit area acts on a triangular segment defined by a hinge and yield lines, such as Fig. 14.8*c*,

$$W_e = \frac{wab}{2} \times \frac{1}{3} = \frac{wab}{6} \tag{c}$$

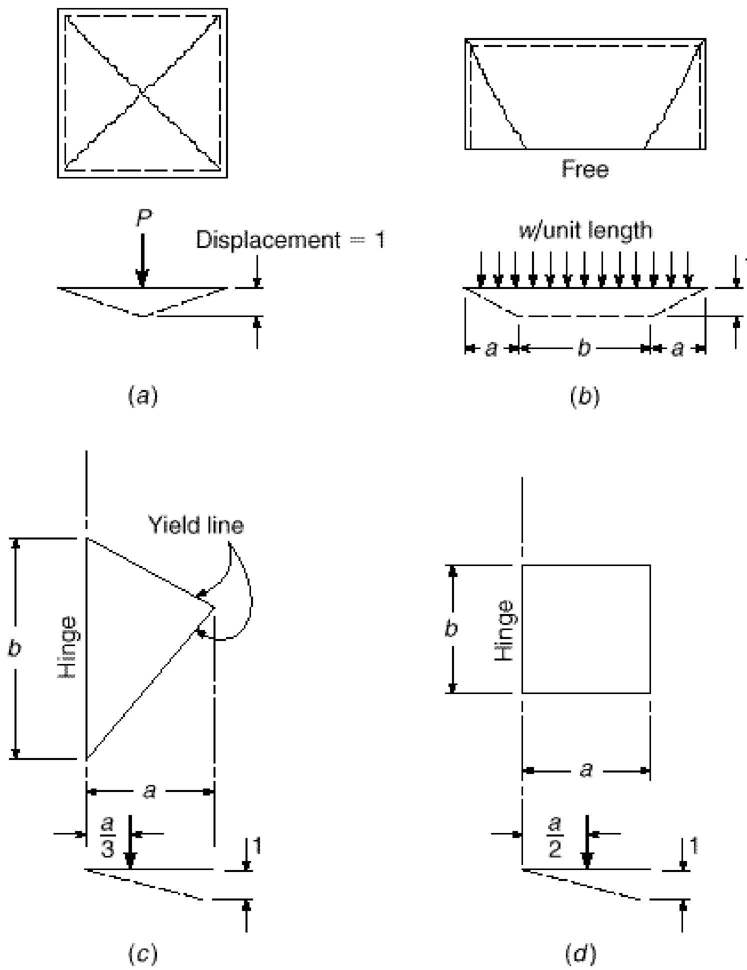


FIGURE 14.8
External work basis for various types of loads.

while for the rectangular slab segment shown in Fig. 14.8*d*, carrying a distributed load w per unit area, the external work is

$$W_e = \frac{wab}{2} \tag{d}$$

More complicated trapezoidal shapes may always be subdivided into component triangles and rectangles. The total external work is then calculated by summing the work done by loads on the individual parts of the failure mechanism, with all displacements keyed to a unit value assigned somewhere in the system. There is no difficulty in combining the work done by concentrated loads, line loads, and distributed loads, if these act in combination.

b. Internal Work Done by Resisting Moments

The internal work done during the assigned virtual displacement is found by summing the products of yield moment m per unit length of hinge times the plastic rotation · at the respective yield lines, consistent with the virtual displacement. If the resisting moment m is constant along a yield line of length l, and if a rotation θ is experienced, the internal work is

$$W_i = ml\theta \tag{e}$$

If the resisting moment varies, as would be the case if bar size or spacing is not constant along the yield line, the yield line is divided into n segments, within each one of which the moment is constant. The internal work is then

$$W_i = (m_1 l_1 + m_2 l_2 + \cdots + m_n l_n)\theta \tag{f}$$

For the entire system, the total internal work done is the sum of the contributions from all yield lines. In all cases, the internal work contributed is positive, regardless of the sign of m, because the rotation is in the same direction as the moment. External work, on the other hand, may be either positive or negative, depending on the direction of the displacement of the point of application of the force resultant.

EXAMPLE 14.3

Virtual work analysis of one-way slab. Determine the load capacity of the one-way uniformly loaded continuous slab shown in Fig. 14.9, using the method of virtual work. The resisting moments of the slab are 5.0, 5.0, and 7.5 ft-kips/ft at A, B, and C, respectively.

Solution. A unit deflection is given to the slab at B. Then the external work done by the load is the sum of the loads times their displacements and is equal to

$$\frac{wx}{2} + \frac{w}{2}(10 - x)$$

The rotations at the hinges are calculated in terms of the unit deflection (Fig. 14.9) and are

$$\theta_A = \theta_{B1} = \frac{1}{x} \qquad \theta_{B2} = \theta_C = \frac{1}{10 - x}$$

The internal work is the sum of the moments times their corresponding rotation angles:

$$5 \times \frac{1}{x} \times 2 + 5 \times \frac{1}{10 - x} + 7.5 \times \frac{1}{10 - x}$$

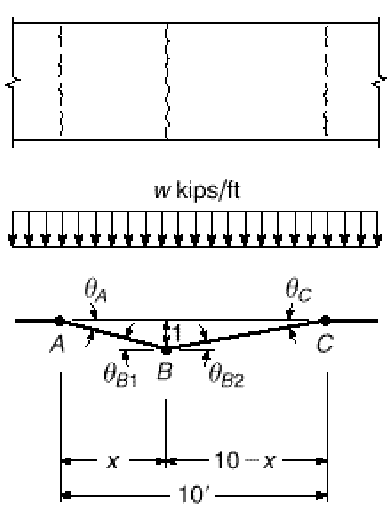


FIGURE 14.9
Virtual work analysis of one-way slab.

Equating the external and internal work gives

$$\frac{wx}{2} + 5w - \frac{wx}{2} = \frac{10}{x} + \frac{5}{10 - x} + \frac{7.5}{10 - x}$$

$$5w = \frac{10}{x} + \frac{25}{2(10 - x)}$$

$$w = \frac{2}{x} + \frac{5}{2(10 - x)}$$

To determine the minimum value of w, this expression is differentiated with respect to x and set equal to zero:

$$\frac{dw}{dx} = -\frac{2}{x^2} + \frac{5}{2(10 - x)^2} = 0$$

from which

$$x = 4.75 \text{ ft}$$

Substituting this value in the preceding expression for w, one obtains

$$w = 0.89 \text{ kips/ft}^2$$

as before.

In many cases, particularly those with yield lines established by several unknown dimensions (such as Fig. 14.4f), direct solution by virtual work would become quite tedious. The ordinary derivatives in Example 14.3 would be replaced by several partial derivatives, producing a set of equations to be solved simultaneously. In such cases it is often more convenient to select an arbitrary succession of possible yield line locations, solve the resulting mechanisms for the unknown load (or unknown moment), and determine the correct minimum load (or maximum moment) by trial.

EXAMPLE 14.4

Virtual work analysis of rectangular slab. The two-way slab shown in Fig. 14.10 is simply supported on all four sides and carries a uniformly distributed load of w psf. Determine the required moment resistance for the slab, which is to be isotropically reinforced.

SOLUTION. Positive yield lines will form in the pattern shown in Fig. 14.10a, with the dimension a unknown. The correct dimension a will be such as to maximize the moment resistance required to support the load w. The values of a and m will be found by trial.

In Fig. 14.10a the length of the diagonal yield line is $\sqrt{25 + a^2}$. From similar triangles,

$$b = 5\frac{\sqrt{25 + a^2}}{a} \qquad c = a\frac{\sqrt{25 + a^2}}{5}$$

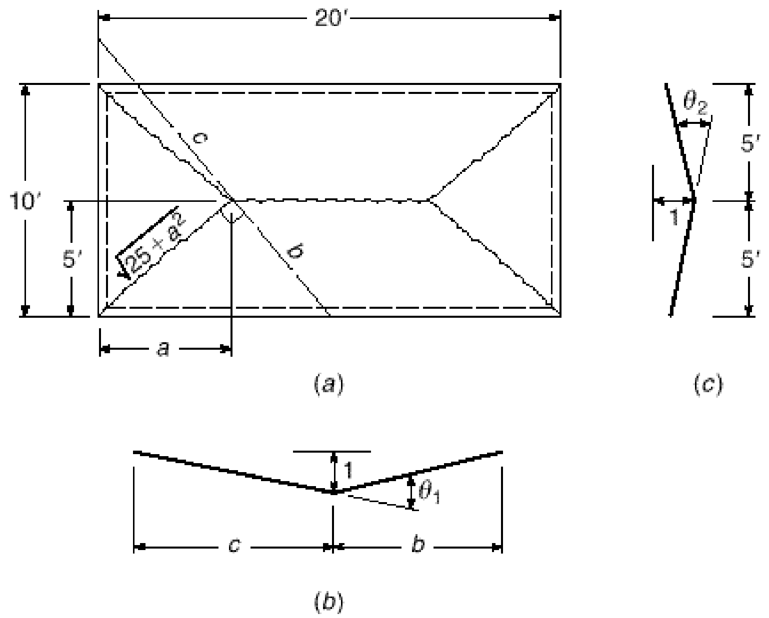
Then the rotation of the plastic hinge at the diagonal yield line corresponding to a unit deflection at the center of the slab (see Fig. 14.10b) is

$$\theta_1 = \frac{1}{b} + \frac{1}{c} = \frac{a}{5\sqrt{25 + a^2}} + \frac{5}{a\sqrt{25 + a^2}} = \frac{1}{\sqrt{25 + a^2}}\left(\frac{a}{5} + \frac{5}{a}\right)$$

The rotation of the yield line parallel to the long edges of the slab (see Fig. 14.10c) is

$$\theta_2 = \frac{1}{5} + \frac{1}{5} = 0.40$$

FIGURE 14.10
Virtual work analysis for rectangular two-way slab.



For a first trial, let $a = 6$ ft. Then the length of the diagonal yield line is

$$\sqrt{25 + 36} = 7.81 \text{ ft}$$

The rotation at the diagonal yield line is

$$\theta_1 = \frac{1}{7.81}\left(\frac{6}{5} + \frac{5}{6}\right) = 0.261$$

At the central yield line, it is $\theta_2 = 0.40$. The internal work done as the incremental deflection is applied is

$$W_i = (m \times 7.81 \times 0.261 \times 4) + (m \times 8 \times 0.40) = 11.36m$$

The external work done during the same deflection is

$$W_e = \left(10 \times 6 \times \tfrac{1}{2}w \times \tfrac{1}{3} \times 2\right) + \left(8 \times 5w \times \tfrac{1}{2} \times 2\right) + \left(12 \times 5 \times \tfrac{1}{2}w \times \tfrac{1}{3} \times 2\right) = 80w$$

Equating W_i and W_e, one obtains

$$m = \frac{80w}{11.36} = 7.05w$$

Successive trials for different values of a result in the following data:

6.0	11.36m	80.0w	7.05w
6.5	11.08m	78.4w	7.08w
7.0	10.87m	76.6w	7.04w
7.5	10.69m	75.0w	7.02w

It is evident that the yield line pattern defined by $a = 6.5$ ft is critical. The required resisting moment for the given slab is 7.08w.

14.6 Orthotropic Reinforcement and Skewed Yield Lines

Generally slab reinforcement is placed orthogonally, i.e., in two perpendicular directions. The same reinforcement is often provided in each direction, but the effective depths will be different. In many practical cases, economical designs are obtained using reinforcement having different bar areas or different spacings in each direction. In such cases, the slab will have different moment capacities in the two orthogonal directions and is said to be orthogonally anisotropic, or simply orthotropic.

Often yield lines will form at an angle with the directions established by the reinforcement; this was so in many of the examples considered earlier. For yield line analysis, it is necessary to calculate the resisting moment, per unit length, along such skewed yield lines. This requires calculation of the contribution to resistance from each of the two sets of bars.

Figure 14.11*a* shows an orthogonal grid of reinforcement, with angle · between the yield line and the *X* direction bars. Bars in the *X* direction are at spacing v and have moment resistance m_y per unit length about the *Y* axis, while bars in the *Y* direction are at spacing u and have moment resistance m_x per unit length about the *X* axis. The resisting moment per unit length for the bars in the *Y* and *X* directions will be determined separately, with reference to Figs. 14.11*b* and *c*, respectively.

For the *Y* direction bars, the resisting moment *per bar* about the *X* axis is $m_x u$, and the component of that resistance about the · axis is $m_x u \cos \cdot$. The resisting moment per unit length along the · axis provided by the *Y* direction bars is therefore

$$m_{\cdot y} = \frac{m_x u \cos \cdot}{u \cdot \cos \cdot} = m_x \cos^2 \cdot \qquad (a)$$

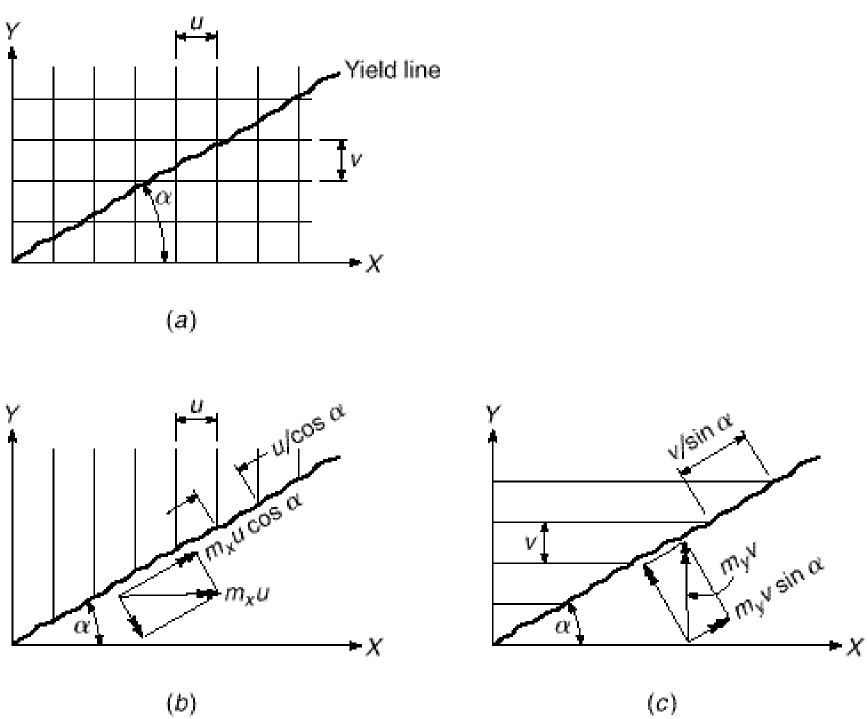


FIGURE 14.11
Yield line skewed with orthotropic reinforcement: (*a*) orthogonal grid and yield line; (*b*) *Y* direction bars; (*c*) *X* direction bars.

For the bars in the X direction, the resisting moment per bar about the Y axis is $m_y v$, and the component of that resistance about the α axis is $m_y v \sin\alpha$. Thus the resisting moment per unit length along the α axis provided by the X direction bars is

$$m_{\alpha x} = \frac{m_y v \sin\alpha}{v / \sin\alpha} = m_y \sin^2\alpha \qquad (b)$$

Thus, for the combined sets of bars, the resisting moment per unit length measured along the α axis is given by the sum of the resistances from Eqs. (a) and (b):

$$m_\alpha = m_x \cos^2\alpha + m_y \sin^2\alpha \qquad (14.1)$$

For the special case where $m_x = m_y = m$, with the same reinforcement provided in each direction,

$$m_\alpha = m(\cos^2\alpha + \sin^2\alpha) = m \qquad (14.2)$$

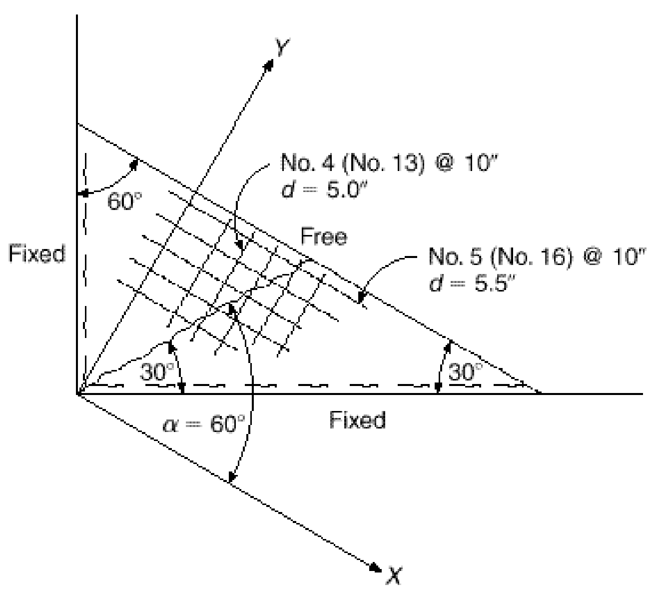
The slab is said to be *isotropically reinforced*, with the same resistance per unit length regardless of the orientation of the yield line.

The analysis just presented neglects any consideration of strain compatibility along the yield line, and assumes that the displacements at the level of the steel during yielding, which are essentially perpendicular to the yield line, are sufficient to produce yielding in both sets of bars. This is reasonably in accordance with test data, except for values of α close to 0 to 90°. For such cases, it would be conservative to neglect the contribution of the bars nearly parallel to the yield line.

It has been shown that the analysis of an orthotropic slab can be simplified to that of a related isotropic slab, referred to as the *affine slab*, provided that the ratio of negative to positive reinforcement areas is the same in both directions. The horizontal dimensions and slab loads must be modified to permit this transformation. Details will be found in Refs. 14.1 to 14.5.

EXAMPLE 14.5 **Resisting moment along a skewed yield line.** The balcony slab in Fig. 14.12 has fixed supports along two adjacent sides and is unsupported along the third side. It is reinforced for positive bending with No. 5 (No. 16) bars at 10 in. spacing and 5.5 in. effective depth, par-

FIGURE 14.12
Skewed yield line example.



allel to the free edge, and No. 4 (No. 13) bars at 10 in. spacing and 5.0 in. effective depth perpendicular to that edge. Concrete strength and steel yield stress are 4000 psi and 60,000 psi, respectively. One possible failure mechanism includes a positive yield line at 30° with the long edge, as shown. Find the total resisting moment along the positive yield line provided by the two sets of bars.

Solution. It is easily confirmed that the resisting moment about the X axis provided by the Y direction bars is $m_x = 5.21$ ft-kips/ft, and the resisting moment about the Y axis provided by the X direction bars is $m_y = 8.70$ ft-kips/ft (both with · = 0.90 included). The yield line makes an angle of 60° with the X axis bars. With cos · = 0.500 and sin · = 0.866, from Eq. (14.1) the resisting moment along the · axis is

$$m_{\cdot} = 5.21 \times 0.500^2 + 8.70 \times 0.866^2 = 7.83 \text{ ft-kips/ft}$$

14.7 Special Conditions at Edges and Corners

Certain simplifications were made in defining yield line patterns in some of the preceding examples, in the vicinity of edges and corners. In some cases, such as Fig. 14.4*b* and *f*, positive yield lines were shown intersecting an edge at an angle. Actually, at a free or simply supported edge, both bending and twisting moments should theoretically be zero. The principal stress directions are parallel and perpendicular to the edge, and consequently the yield lines should enter an edge perpendicular to it. Tests confirm that this is the case, but the yield lines generally turn only quite close to the edge, the distance t in Fig. 14.13 being small compared to the dimensions of the slab (Ref. 14.4).

Referring to Fig. 14.13, the actual yield line of *a* can be simplified by extending the yield line in a straight line to the edge, as in *b*, if a pair of concentrated shearing forces m_t is introduced at the corners of the slab segments. The force m_t acting downward at the acute corner (circled cross) and the force m_t acting upward at the obtuse corner (circled dot) together are the static equivalent of twisting moments and shearing forces near the edge. It is shown in Ref. 14.4 that the magnitude of the fictitious shearing forces m_t is given by the expression

$$m_t = m \cot \cdot \tag{14.3}$$

where m is the resisting moment per unit length along the yield line and · is the acute angle between the simplified yield line and the edge of the slab.

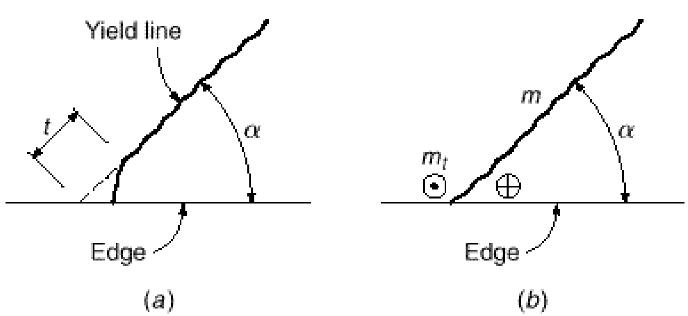
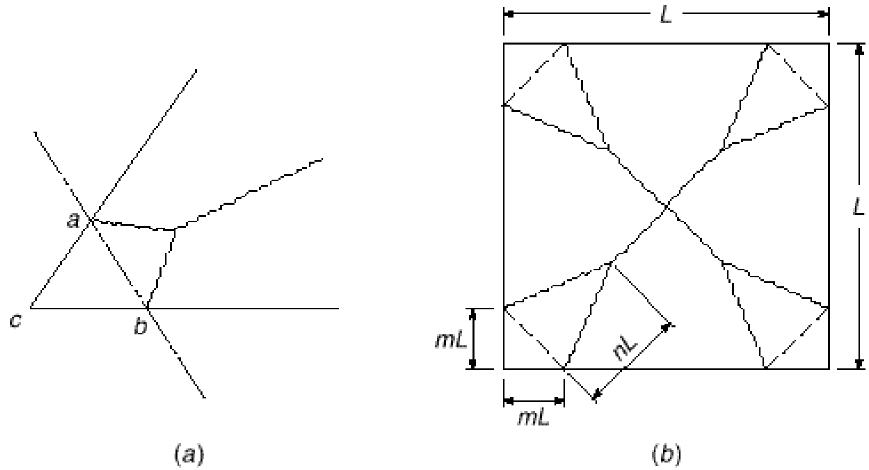


FIGURE 14.13
Conditions at edge of slab: (*a*) actual yield line; (*b*) simplified yield line.

FIGURE 14.14
Corner conditions.

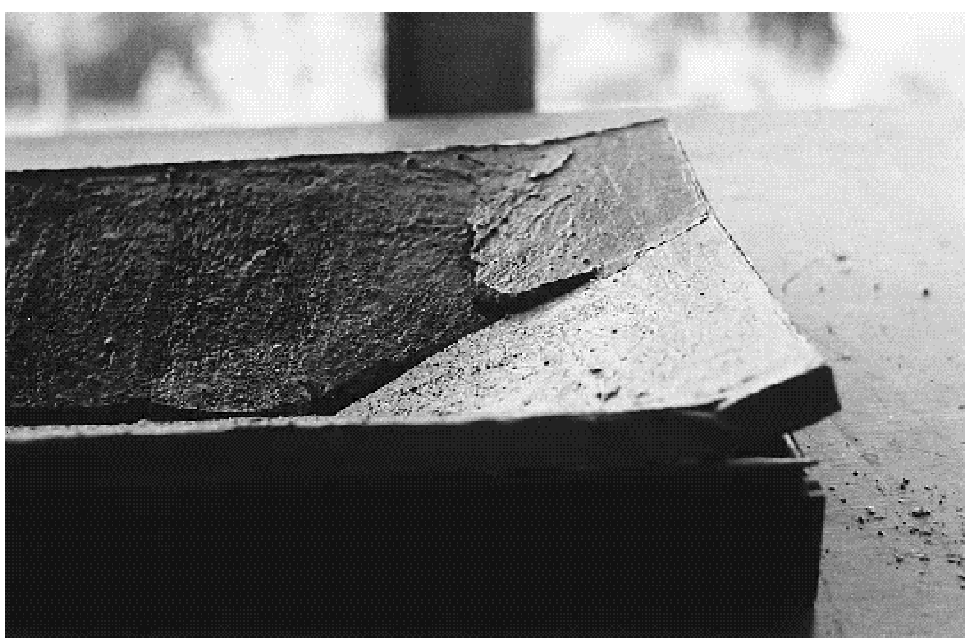


It should be noted that, while the fictitious forces enter the solution by the equilibrium method, the virtual work solution is not affected because the net work done by the pair of equal and opposite forces moving through the identical virtual displacement is zero.

Also, in the preceding examples, it was assumed that yield lines enter the corners between the two intersecting sides. An alternative possibility is that the yield line forks before it reaches the corner, forming what is known as a *corner lever*, shown in Fig. 14.14*a*.

If the corner is not held down, the triangular element *abc* will pivot about the axis *ab* and lift off the supports. The development of such a corner lever is clearly shown in Fig. 14.15. The photograph shows a model reinforced concrete slab that was tested under uniformly distributed load. The edges were simply supported and were

FIGURE 14.15
Development of corner levers in a simply supported, uniformly loaded slab.



not restrained against upward movement. If the corner is held down, a similar situation occurs, except that the line *ab* becomes a yield line. If cracking at the corners of such a slab is to be controlled, top steel, more or less perpendicular to the line *ab* must be provided. The direction taken by the positive yield lines near the corner indicates the desirability of supplementary bottom-slab reinforcement at the corners, placed approximately parallel to the line *ab* (see Section 13.4).

Although yield line patterns with corner levers are generally more critical than those without, they are often neglected in yield line analysis. The analysis becomes considerably more complicated if the possibility of corner levers is introduced, and the error made by neglecting them is usually small.

To illustrate, the uniformly loaded square slab of Example 14.2, when analyzed for the assumed yield pattern shown in Fig. 14.7, required a moment capacity of $wL^2/24$. The actual yield line pattern at failure is probably as shown in Fig. 14.14*b*. Since two additional parameters m and n have necessarily been introduced to define the yield line pattern, a total of three equations of equilibrium is now necessary. These equations are obtained by summing moments and vertical forces on the segments of the slab. Such an analysis results in a required resisting moment of $wL^2/22$, an increase of about 9 percent compared with the results of an analysis neglecting corner levers. The influence of such corner effects may be considerably larger when the corner angle is less than 90°.

14.8 Fan Patterns at Concentrated Loads

If a concentrated load acts on a reinforced concrete slab at an interior location, away from any edge or corner, a negative yield line will form in a more-or-less circular pattern, as in Fig. 14.16*a*, with positive yield lines radiating outward from the load point.

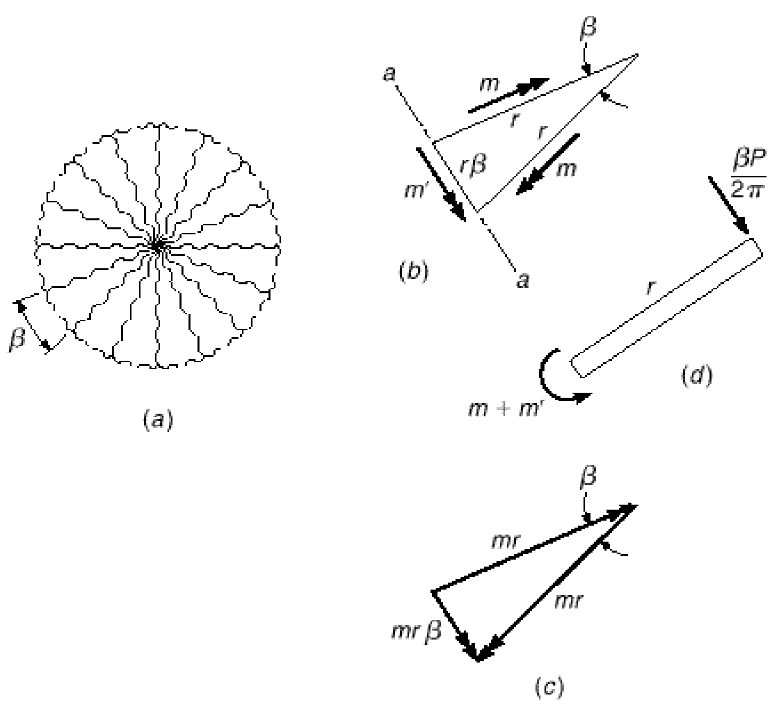


FIGURE 14.16
Yield fan geometry at concentrated load: (*a*) yield fan; (*b*) moment vectors acting on fan segment; (*c*) resultant of positive-moment vectors; (*d*) edge view of fan segment.

If the positive resisting moment per unit length is m and the negative resisting moment m', the moments per unit length acting along the edges of a single element of the fan, having a central angle and radius r, are as shown in Fig. 14.16*b*. For small values of the angle , the arc along the negative yield line can be represented as a straight line of length r .

Figure 14.16*c* shows the moment resultant obtained by vector addition of the positive moments mr acting along the radial edges of the fan segment. The vector sum is equal to mr , acting along the length r , and the resultant positive moment, per unit length, is therefore m. This acts in the same direction as the negative moment m', as shown in Fig. 14.16*d*. Figure 14.16*d* also shows the fractional part of the total load P that acts on the fan segment.

Taking moments about the axis $a - a$ gives

$$m + m \cdot r - \frac{Pr}{2} = 0$$

from which

$$P = 2 (m + m') \tag{14.4}$$

The collapse load P is seen to be independent of the fan radius r. Thus, with only a concentrated load acting, a complete fan of any radius could form with no change in collapse load.

It follows that Eq. (14.4) also gives the collapse load for a fixed-edge slab of any shape, carrying only a concentrated load P. The only necessary condition is that the boundary must be capable of a restraining moment equal to m' at all points. Finally, Eq. (14.4) is useful in establishing whether flexural failure will occur before a punching shear failure under a concentrated load.

Other load cases of practical interest, including a concentrated load near or at a free edge, and a concentrated corner load, are treated in Ref. 14.5. Loads distributed over small areas and load combinations are discussed in Ref. 14.12.

14.9 LIMITATIONS OF YIELD LINE THEORY

The usefulness of yield line theory should be apparent from the preceding sections. In general, elastic solutions are available only for restricted conditions, usually uniformly loaded rectangular slabs and slab systems. They do not account for the effects of inelastic action, except empirically. By yield line analysis, a rational determination of flexural strength may be had for slabs of any shape, supported in a variety of ways, with concentrated loads as well as distributed and partially distributed loads. The effects of holes of any size can be included. It is thus seen to be a powerful analytical tool for the structural engineer.

On the other hand, as an upper bound method, it will predict a collapse load that may be greater than the true collapse load. The actual capacity will be less than predicted if the selected mechanism is not the controlling one or if the specific locations of yield lines are not exactly correct. Most engineers would prefer an approach that would be in error, if at all, on the safe side. In this respect, the strip method of Chapter 15 is distinctly superior.

Beyond this, it should be evident that yield line theory provides, in essence, a method for determining the capacity of trial designs, arrived at by some other means, rather than for determining the amount and spacing of reinforcement. It is not, strictly speaking, a design method. To illustrate, yield line theory provides no inducement for the designer to place steel at anything other than a uniform lateral spacing along a yield line. It is necessary to consider the results of elastic analysis of a flat plate, for example, to recognize that reinforcement in that case should be placed in strong bands across the columns.

In applying yield line analysis to slabs, it must be remembered that the analysis is predicated upon available rotation capacity at the yield lines. If the slab reinforcement happens to correspond closely to the elastic distribution of moments in the slab, little rotation is required. If, on the other hand, there is a marked difference, it is possible that the required rotation will exceed the available rotation capacity, in which case the slab will fail prematurely. However, in general, because slabs are typically rather lightly reinforced, they will have adequate rotation capacity to attain the collapse loads predicted by yield line analysis.

It should also be borne in mind that the yield line analysis focuses entirely on the flexural capacity of the slab. It is presumed that earlier failure will not occur due to shear or torsion and that cracking and deflections at service load will not be excessive. ACI Code 13.5.1 calls attention specifically to the need to meet "all serviceability conditions, including limits on deflections," and ACI Commentary R13.5.1 calls attention to the need for "evaluation of the stress conditions around the supports in relation to shear and torsion as well as flexure."

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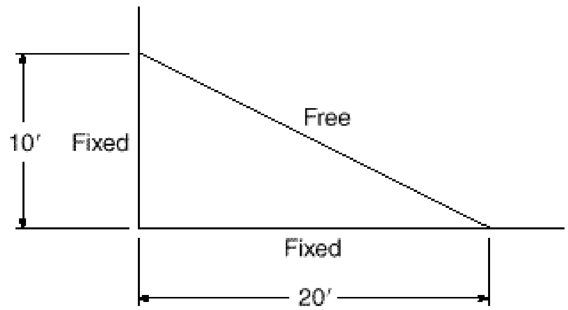
PROBLEMS

14.1. A square slab measuring 10 ft on each side is simply supported on three sides and unsupported along the fourth. It is reinforced for positive bending with an isotropic mat of steel providing resistance · m_n of 7000 ft-lb/ft in each of the two principal directions. Determine the uniformly distributed load that would cause flexural failure, using the method of virtual work.

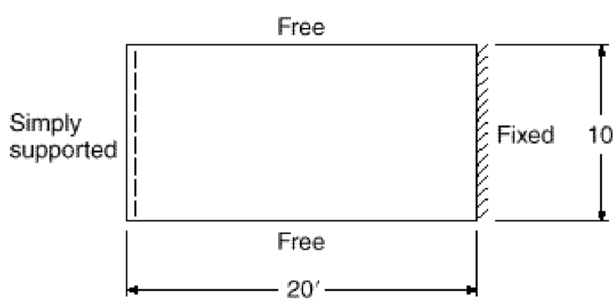
14.2. The triangular slab shown in Fig. P14.2 has fixed supports along the two perpendicular edges and is free of any support along the diagonal edge. Negative reinforcement perpendicular to the supported edges provides design strength · m_n = 4 ft-kips/ft. The slab is reinforced for positive bending by an orthogonal grid providing resistance · m_n = 2.67 ft-kips/ft in all directions. Find the total factored load w_u that will produce flexural failure. A virtual work solution is suggested.

FIGURE P14.2



14.3. The one-way reinforced concrete slab shown in Fig. P14.3 spans 20 ft. It is simply supported at its left edge, fully fixed at its right edge, and free of support along the two long sides. Reinforcement provides design strength · m_n = 5 ft-kips/ft in positive bending and · m_n = 7.5 ft-kips/ft in negative bending at the right edge. Using the equilibrium method, find the factored load w_u uniformly distributed over the surface that would cause flexural failure.

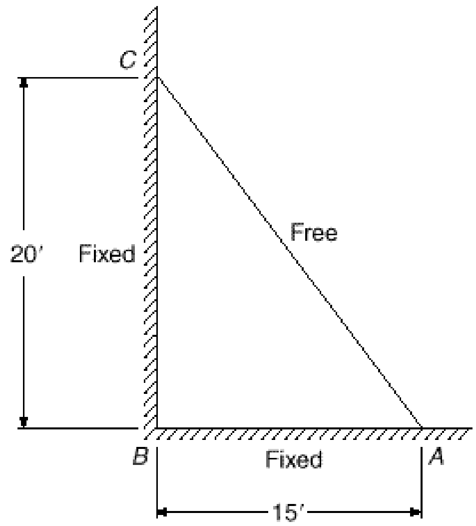
FIGURE P14.3



14.4. Solve Problem 14.3 using the method of virtual work.

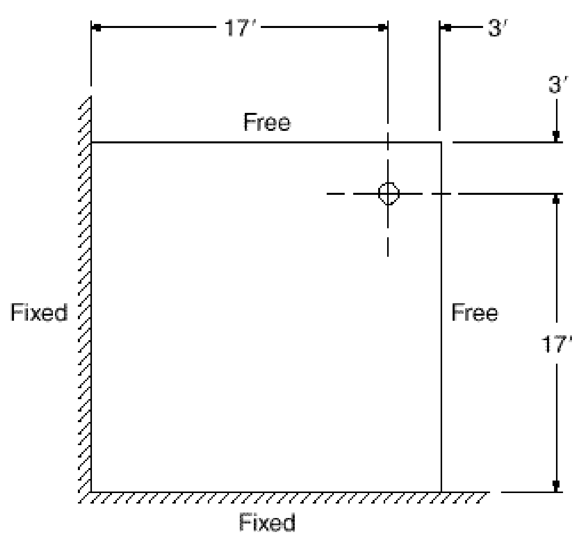
14.5. The triangular slab shown in Fig. P14.5 is to serve as weather protection over a loading dock. Support conditions are essentially fixed along *AB* and *BC*, and *AC* is a free edge. In addition to self-weight, a superimposed dead load of 15 psf and service live load of 40 psf must be provided for. Material strengths are f'_c = 4000 psi and f_y = 60,000 psi. Using yield line analysis, find the required slab thickness *h* and find the reinforcement required at critical sections. Neglect corner pivots. Use a maximum reinforcement ratio of 0.005. Select bar sizes and spacings, and provide a sketch summarizing important aspects of the design. Make an approximate, conservative check of safety against shear failure for the design. Also include a conservative estimate of the deflection near the center of edge *AC* due to a full live load.

FIGURE P14.5



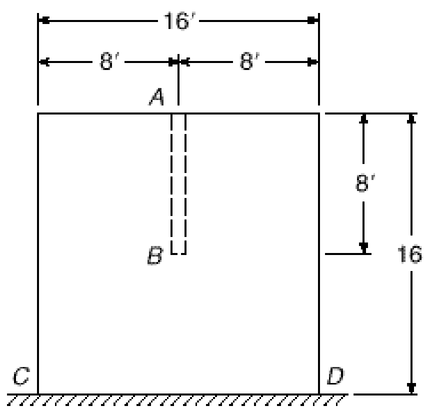
14.6. The square concrete slab shown in Fig. P14.6 is supported by monolithic concrete walls providing full vertical and rotational restraint along two adjacent edges, and by a 6 in. diameter steel pipe column, near the outer corner, that offers negligible rotational restraint. It is reinforced for positive bending by an orthogonal grid of bars parallel to the walls, providing design moment capacity $\cdot\, m_n = 6.5$ ft-kips/ft in all directions. Negative reinforcement perpendicular to the walls, and negative bars at the outer corner parallel to the slab diagonal, provide $\cdot\, m_n = 8.9$ ft-kips/ft. Neglecting corner pivots, find the total factored uniformly distributed load w_u that will initiate flexural failure. Solution by the method of virtual work is recommended, with collapse geometry established by successive trial. Yield line lengths and perpendicular distances are most easily found graphically. Include a check of the shear capacity of the slab, using approximate methods. The steel column is capped with a 12 × 12 in. plate providing bearing.

FIGURE P14.6



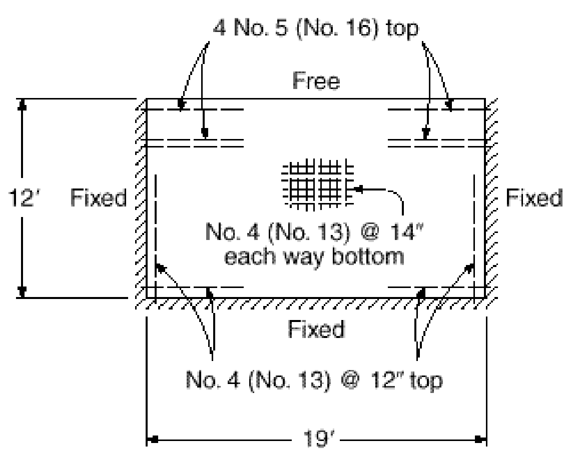
14.7. The square slab shown in Fig. P14.7 is supported by, and is monolithic with, a reinforced concrete wall along the edge *CD* that provides full fixity, and also is supported by a masonry wall along *AB* that provides a simply supported line. It is to carry a factored load w_u = 300 psf including its self-weight. Assuming a uniform 6 in. slab thickness, find the required reinforcement. Include a sketch summarizing details of your design, indicating placement and length of all reinforcing bars. Also check the shear capacity of the structure, making whatever assumptions appear reasonable and necessary. Use f'_c = 4000 psi and f_y = 60,000 psi.

FIGURE P14.7



14.8. The slab of Fig. P14.8 is supported by three fixed edges but has no support along one long side. It has a uniform thickness of 7 in., resulting in effective depths in the long direction of 6.0. in. and in the short direction of 5.5 in. Bottom reinforcement consists of No. 4 (No. 13) bars at 14 in. centers in each direction, continued to the supports and the free edge. Top negative steel along the supported edges consists of No. 4 (No. 13) bars at 12 in. on centers, except that in a 2 ft wide "strong band" parallel and adjacent to the free edge, four No. 5 (No. 16) bars are used. All negative bars extend past the points of inflection, as required by ACI Code. Material strengths are f'_c = 4000 psi and f_y = 60,000 psi. Using the yield line method, determine the factored load w_u that can be carried.

FIGURE P14.8



14.9. Using virtual work and yield line theory, compute the flexural collapse load of the one-way slab in Example 13.1. Assume that all straight bars are used, according to Fig. 13.4*b*. Compare the calculated collapse load with the original factored design load, and comment on differences.

14.10. Using virtual work and yield line theory, compute the flexural collapse load of the two-way column-supported flat plate of Example 13.3. To simplify the calculations, assume that all positive moment bars are carried to the edges of the panels, not cut off in the span. Consider all possible failure mechanisms, including a circular fan around the column. Neglect corner effects. Compare the calculated collapse load with the original factored design load and comment on differences.