15 Strip Method for Slabs

15.1 Introduction

In Section 14.2, the upper and lower bound theorems of the theory of plasticity were presented, and it was pointed out that the yield line method of slab analysis was an *upper bound approach* to determining the flexural strength of slabs. An upper bound analysis, if in error, will be so on the unsafe side. The actual carrying capacity will be less than, or at best equal to, the capacity predicted, which is certainly a cause for concern in design. Also, when applying the yield line method, it is necessary to assume that the distribution of reinforcement is known over the whole slab. It follows that the yield line approach is a tool to *analyze* the capacity of a given slab and can be used for *design* only in an iterative sense, for calculating the capacities of trial designs with varying reinforcement until a satisfactory arrangement is found.

These circumstances motivated Hillerborg to develop what is known as the *strip method* for slab design, his first results being published in Swedish in 1956 (Ref. 15.1). In contrast to yield line analysis, the strip method is a *lower bound approach*, based on satisfaction of equilibrium requirements everywhere in the slab. By the strip method (sometimes referred to as the *equilibrium theory*), a moment field is first determined that fulfills equilibrium requirements, after which the reinforcement in the slab at each point is designed for this moment field. If a distribution of moments can be found that satisfies both equilibrium and boundary conditions for a given external loading, and if the yield moment capacity of the slab is nowhere exceeded, then the given external loading will represent a lower bound of the true carrying capacity.

The strip method gives results on the safe side, which is certainly preferable in practice, and differences from the true carrying capacity will never impair safety. The strip method is a *design* method, by which the needed reinforcement can be calculated. It encourages the designer to vary the reinforcement in a logical way, leading to an economical arrangement of steel, as well as a safe design. It is generally simple to use, even for slabs with holes or irregular boundaries.

In his original work in 1956, Hillerborg set forth the basic principles for edge-supported slabs and introduced the expression "strip method" (Ref. 15.1). He later expanded the method to include the practical design of slabs on columns and L-shaped slabs (Refs. 15.2 and 15.3). The first treatment of the subject in English was by Crawford (Ref. 15.4). In 1964, Blakey translated the earlier Hillerborg work into English (Ref. 15.5). Important contributions, particularly regarding continuity conditions, have been made by Kemp (Refs. 15.6 and 15.7) and Wood and Armer (Refs. 15.8, 15.9, and 15.10). Load tests of slabs designed by the strip method were carried out by Armer (Ref. 15.11) and confirmed that the method produces safe and satisfactory designs. In 1975, Hillerborg produced Ref. 15.12 "for the practical designer, helping him in the simplest possible way to produce safe designs for most of the slabs that he will meet in practice, including slabs that are irregular in plan or that carry unevenly distributed loads." Subsequently, he published a paper in which he summarized what has become known as the "advanced strip method," pertaining to the design of slabs supported on columns, reentrant corners, or interior walls (Ref. 15.13). Useful summaries of both the simple and advanced strip methods will be found in Refs. 15.14 and 15.15.

The strip method is appealing not only because it is safe, economical, and versatile over a broad range of applications, but also because it formalizes procedures followed instinctively by competent designers in placing reinforcement in the best possible position. In contrast with the yield line method, which provides no inducement to vary bar spacing, the strip method encourages the use of strong bands of steel where needed, such as around openings or over columns, improving economy and reducing the likelihood of excessive cracking or large deflections under service loading.

15.2 Basic Principles

The governing equilibrium equation for a small slab element having sides dx and dy is

$$\frac{\partial^2 m_x}{\partial x^2} + \frac{\partial^2 m_y}{\partial y^2} - 2\frac{\partial^2 m_{xy}}{\partial x \partial y} = -w \tag{15.1}$$

where w = external load per unit area
m_x, m_y = bending moments per unit width in X and Y directions, respectively
m_{xy} = twisting moment (Ref. 15.16).

According to the lower bound theorem, any combination of m_x, m_y, and m_{xy} that satisfies the equilibrium equation at all points in the slab and that meets boundary conditions is a valid solution, provided that the reinforcement is placed to carry these moments.

The basis for the simple strip method is that the torsional moment is chosen equal to zero; no load is assumed to be resisted by the twisting strength of the slab. Therefore, if the reinforcement is parallel to the axes in a rectilinear coordinate system,

$$m_{xy} = 0$$

The equilibrium equation then reduces to

$$\frac{\partial^2 m_x}{\partial x^2} + \frac{\partial^2 m_y}{\partial y^2} = -w \tag{15.2}$$

This equation can be split conveniently into two parts, representing twistless beam strip action,

$$\frac{\partial^2 m_x}{\partial x^2} = -kw \tag{15.3a}$$

and

$$\frac{\partial^2 m_y}{\partial y^2} = -(1-k)w \tag{15.3b}$$

where the proportion of load taken by the strips is k in the X direction and $(1 - k)$ in the Y direction. In many regions in slabs, the value of k will be either 0 or 1. With $k = 0$, all of the load is dispersed by strips in the Y direction; with $k = 1$, all of the load is carried in the X direction. In other regions, it may be reasonable to assume that the load is divided equally in the two directions (i.e., $k = 0.5$).

15.3 Choice of Load Distribution

Theoretically, the load w can be divided arbitrarily between the X and Y directions. Different divisions will, of course, lead to different patterns of reinforcement, and all will not be equally appropriate. The desired goal is to arrive at an arrangement of steel that is safe and economical and that will avoid problems at the service load level associated with excessive cracking or deflections. In general, the designer may be guided by knowledge of the general distribution of elastic moments.

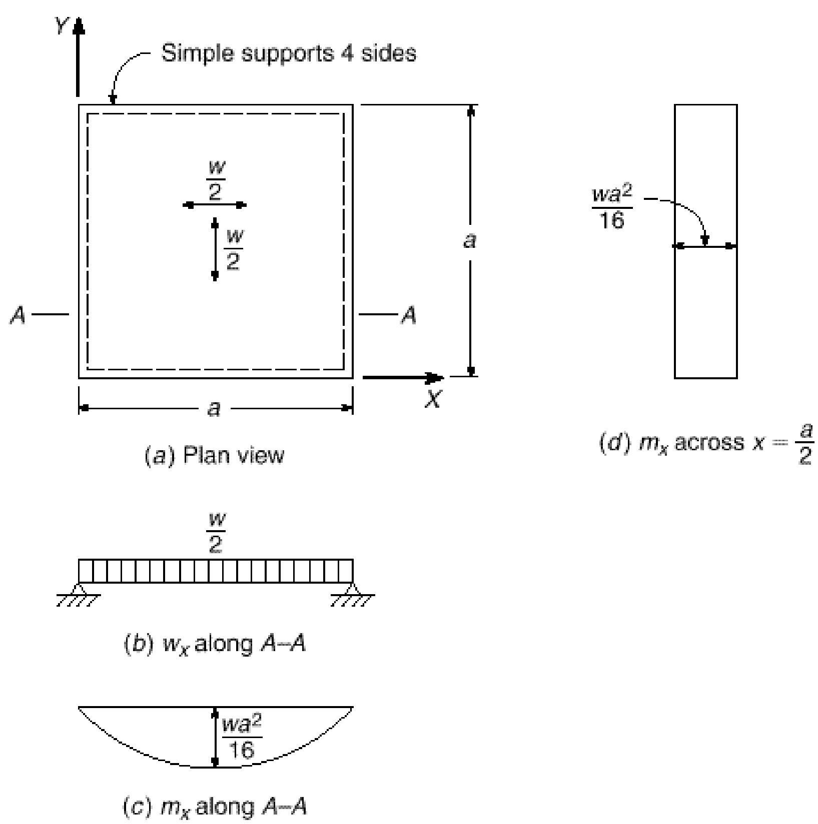
To see an example of the strip method and to illustrate the choices open to the designer, consider the square, simply supported slab shown in Fig. 15.1, with side length a and a uniformly distributed factored load w per unit area.

The simplest load distribution is obtained by setting $k = 0.5$ over the entire slab, as shown in Fig. 15.1. The load on all strips in each direction is then $w/2$, as illustrated by the load dispersion arrows of Fig. 15.1*a*. This gives maximum moments

$$m_x = m_y = \frac{wa^2}{16} \tag{15.4}$$

over the whole slab, as shown in Fig. 15.1*c*, with a uniform lateral distribution across the width of the critical section, as in Fig. 15.1*d*.

FIGURE 15.1
Square slab with load shared equally in two directions.



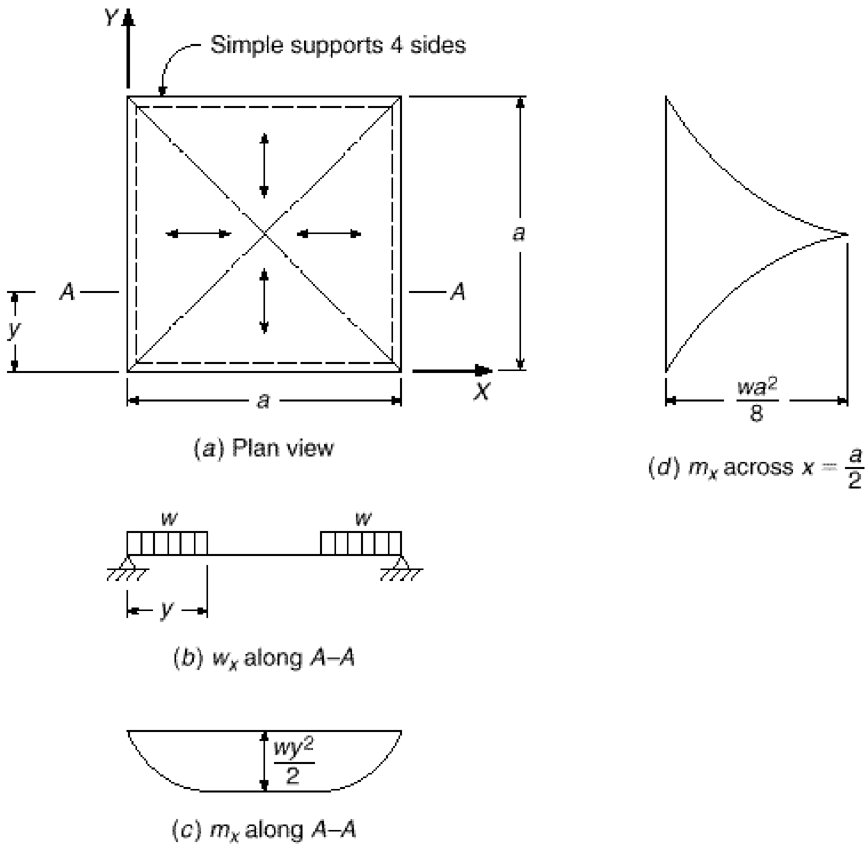


FIGURE 15.2
Square slab with load dispersion lines following diagonals.

This would not represent an economical or serviceable solution because it is recognized that curvatures, hence moments, must be greater in the strips near the middle of the slab than near the edges in the direction parallel to the edge (see Fig. 13.5). If the slab were reinforced according to this solution, extensive redistribution of moments would be required, certainly accompanied by much cracking in the highly stressed regions near the middle of the slab.

An alternative, more reasonable distribution is shown in Fig. 15.2. Here the regions of different load dispersion, separated by the dash-dotted "discontinuity lines," follow the diagonals, and all of the load on any region is carried in the direction giving the shortest distance to the nearest support. The solution proceeds, giving k values of either 0 or 1, depending on the region, with load transmitted in the directions indicated by the arrows of Fig. 15.2*a*. For a strip *A-A* at a distance $y \leq a/2$ from the X axis, the moment is

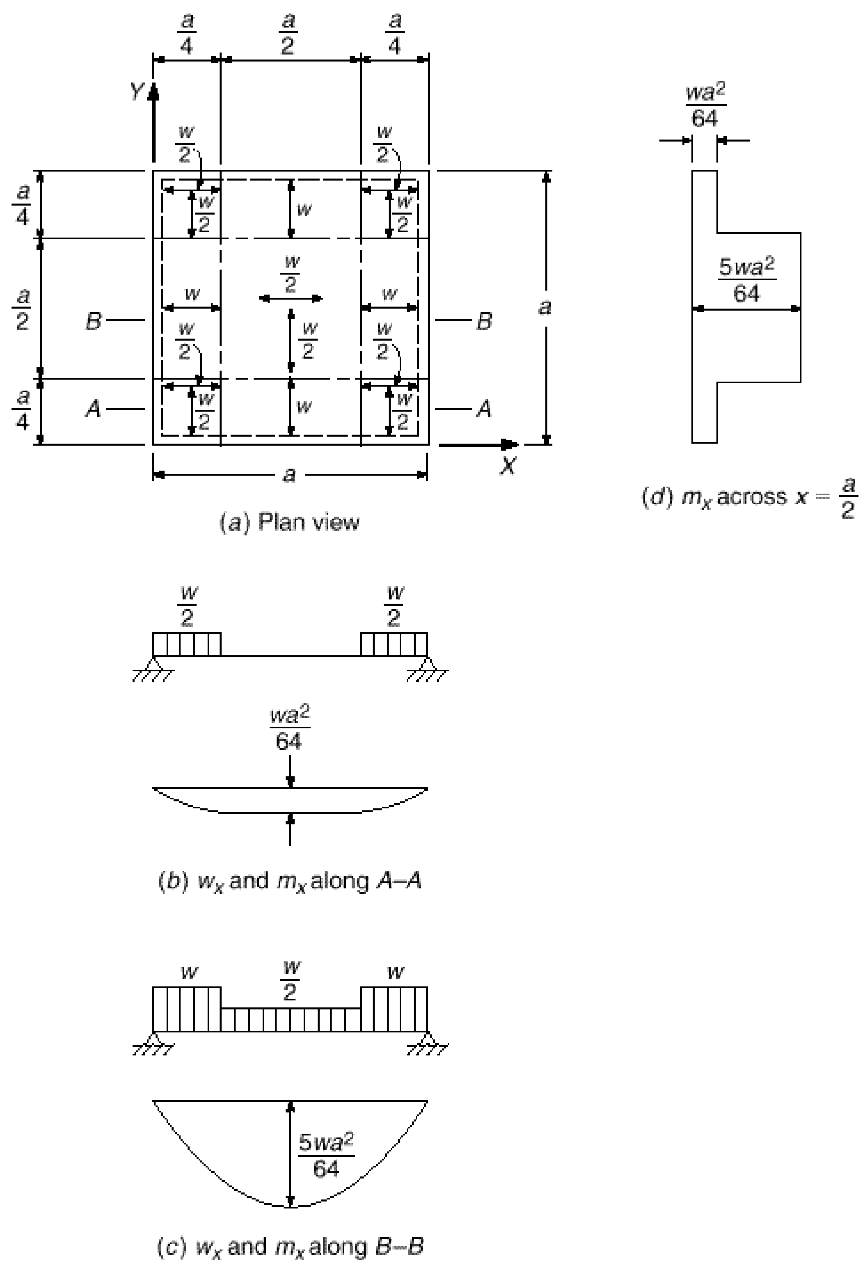
$$m_x = \frac{wy^2}{2} \tag{15.5}$$

The load acting on a strip *A-A* is shown in Fig. 15.2*b*, and the resulting diagram of moment m_x is given in Fig. 15.2*c*. The lateral variation of m_x across the width of the slab is as shown in Fig. 15.2*d*.

The lateral distribution of moments shown in Fig. 15.2*d* would theoretically require a continuously variable bar spacing, obviously an impracticality. One way of using the distribution in Fig. 15.2, which is considerably more economical than that in Fig. 15.1, would be to reinforce for the *average* moment over a certain width, approximating the actual lateral variation shown in Fig. 15.2*d* in a stepwise manner. Hillerborg notes that this is not strictly in accordance with the equilibrium theory and that the design is no longer certainly on the safe side, but other conservative assumptions, e.g., neglect of membrane strength in the slab and neglect of strain hardening of the reinforcement, would surely compensate for the slight reduction in safety margin.

A third alternative distribution is shown in Fig. 15.3. Here the division is made so that the load is carried to the nearest support, as before, but load near the diagonals

FIGURE 15.3
Square slab with load near diagonals shared equally in two directions.



has been divided, with one-half taken in each direction. Thus, k is given values of 0 or 1 along the middle edges and a value of 0.5 in the corners and center of the slab, with load dispersion in the directions indicated by the arrows shown in Fig. 15.3*a*. Two different strip loadings are now identified. For an X direction strip along section A-A, the maximum moment is

$$m_x = \frac{w}{2} \times \frac{a}{4} \times \frac{a}{8} = \frac{wa^2}{64} \tag{15.6a}$$

and for a strip along section B-B, the maximum moment is

$$m_x = w \times \frac{a}{4} \times \frac{a}{8} + \frac{w}{2} \times \frac{a}{4} \times \frac{3a}{8} = \frac{5wa^2}{64} \tag{15.6b}$$

This design leads to a practical arrangement of reinforcement, one with constant spacing through the center strip of width $a/2$ and a wider spacing through the outer strips, where the elastic curvatures and moments are known to be less. The averaging of moments necessitated in the second solution is avoided here, and the third solution is fully consistent with the equilibrium theory.

Comparing the three solutions just presented shows that the first would be unsatisfactory, as noted earlier, because it would require great redistribution of moments to achieve, possibly accompanied by excessive cracking and large deflections. The second, with discontinuity lines following the slab diagonals, has the advantage that the reinforcement more nearly matches the elastic distribution of moments, but it either leads to an impractical reinforcing pattern or requires an averaging of moments in bands that involves a deviation from strict equilibrium theory. The third solution, with discontinuity lines parallel to the edges, does not require moment averaging and leads to a practical reinforcing arrangement, so it will often be preferred.

The three examples also illustrate the simple way in which moments in the slab can be found by the strip method, based on familiar beam analysis. It is important to note, too, that the load on the supporting beams is easily found because it can be computed from the end reactions of the slab beam strips in all cases. This information is not available from solutions such as those obtained by the yield line theory.

15.4 Rectangular Slabs

With rectangular slabs, it is reasonable to assume that, throughout most of the area, the load will be carried in the short direction, consistent with elastic theory (see Section 13.4). In addition, it is important to take into account the fact that because of their length, longitudinal reinforcing bars will be more expensive than transverse bars of the same size and spacing. For a uniformly loaded rectangular slab on simple supports, Hillerborg presents one possible division, as shown in Fig. 15.4, with discontinuity lines originating from the slab corners at an angle depending on the ratio of short to long sides of the slab. All of the load in each region is assumed to be carried in the directions indicated by the arrows.

Instead of the solution of Fig. 15.4, which requires continuously varying reinforcement to be strictly correct, Hillerborg suggests that the load can be distributed as shown in Fig. 15.5, with discontinuity lines parallel to the sides of the slab. For such cases, it is reasonable to take edge bands of width equal to one-fourth the short span dimension. Here the load in the corners is divided equally in the X and Y directions as shown, while elsewhere all of the load is carried in the direction indicated by the arrows.

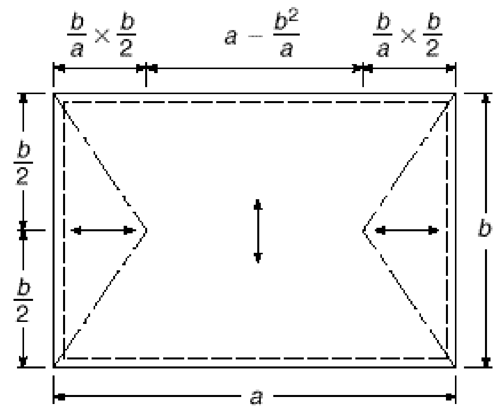


FIGURE 15.4
Rectangular slab with discontinuity lines originating at the corners.

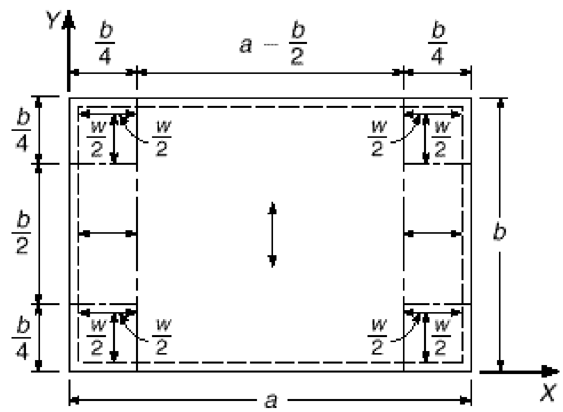


FIGURE 15.5
Discontinuity lines parallel to the sides for a rectangular slab.

The second, preferred arrangement, shown in Fig. 15.5, gives slab moments as follows:

In the X direction:

Side strips:
$$m_x = \frac{w}{2} \times \frac{b}{4} \times \frac{b}{8} = \frac{wb^2}{64} \tag{15.7a}$$

Middle strips:
$$m_x = w \times \frac{b}{4} \times \frac{b}{8} = \frac{wb^2}{32} \tag{15.7b}$$

In the Y direction:

Side strips:
$$m_y = \frac{wb^2}{64} \tag{15.8a}$$

Middle strips:
$$m_y = \frac{wb^2}{8} \tag{15.8b}$$

This distribution, requiring no averaging of moments across band widths, is always on the safe side and is both simple and economical.

15.5 Fixed Edges and Continuity

Designing by the strip method has been shown to provide a large amount of flexibility in assigning load to various regions of slabs. This same flexibility extends to the assignment of moments between negative and positive bending sections of slabs that are fixed or continuous over their supported edges. Some attention should be paid to elastic moment ratios to avoid problems with cracking and deflection at service loads. However, the redistribution that can be achieved in slabs, which are typically rather lightly reinforced and, thus, have large plastic rotation capacities when overloaded, permits considerable arbitrary readjustment of the ratio of negative to positive moments in a strip.

This is illustrated by Fig. 15.6, which shows a slab strip carrying loads only near the supports and unloaded in the central region, such as often occurs in designing by the strip method. It is convenient if the unloaded region is subject to a constant moment (and zero shear), because this simplifies the selection of positive reinforcement. The sum of the absolute values of positive span moment and negative end moment at the left or right end, shown as m_l and m_r in Fig. 15.6, depends only on the conditions at the respective end and is numerically equal to the negative moment if the strip carries the load as a cantilever. Thus, in determining moments for design, one calculates the "cantilever" moments, selects the span moment, and determines the corresponding support moments. Hillerborg notes that, as a general rule for fixed edges, the support moment should be about 1.5 to 2.5 times the span moment in the same strip. Higher values should be chosen for longitudinal strips that are largely unloaded, and in such cases a ratio of support to span moment of 3 to 4 may be used. However, little will be gained by using such a high ratio if the positive moment steel is controlled by minimum requirements of the ACI Code.

For slab strips with one end fixed and one end simply supported, the dual goals of constant moment in the unloaded central region and a suitable ratio of negative to positive moments govern the location to be chosen for the discontinuity lines. Figure 15.7*a* shows a uniformly loaded rectangular slab having two adjacent edges fixed and the other two edges simply supported. Note that, although the middle strips have the same width as those of Fig. 15.5, the discontinuity lines are shifted to account for the greater stiffness of the strips with fixed ends. Their location is defined by a coefficient · , with a value clearly less than 0.5 for the slab shown, its exact value yet to be determined. It will be seen that the selection of · relates directly to the ratio of negative to positive moments in the strips.

The moment curve of Fig. 15.7*b* is chosen so that moment is constant over the unloaded part, i.e., shearing force is zero. With constant moment, the positive steel can be fully stressed over most of the strip. The maximum positive moment in the *X* direction middle strip is then

$$m_{xf} = \frac{\cdot\, wb}{2} \times \frac{\cdot\, b}{4} = \cdot^{2}\,\frac{wb^2}{8} \tag{15.9}$$

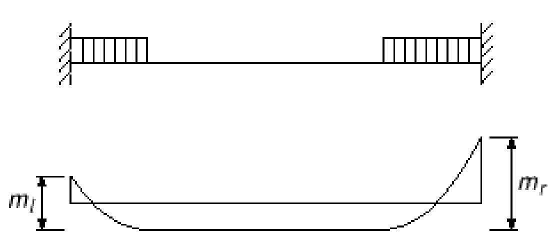


FIGURE 15.6
Slab strip with central region unloaded.

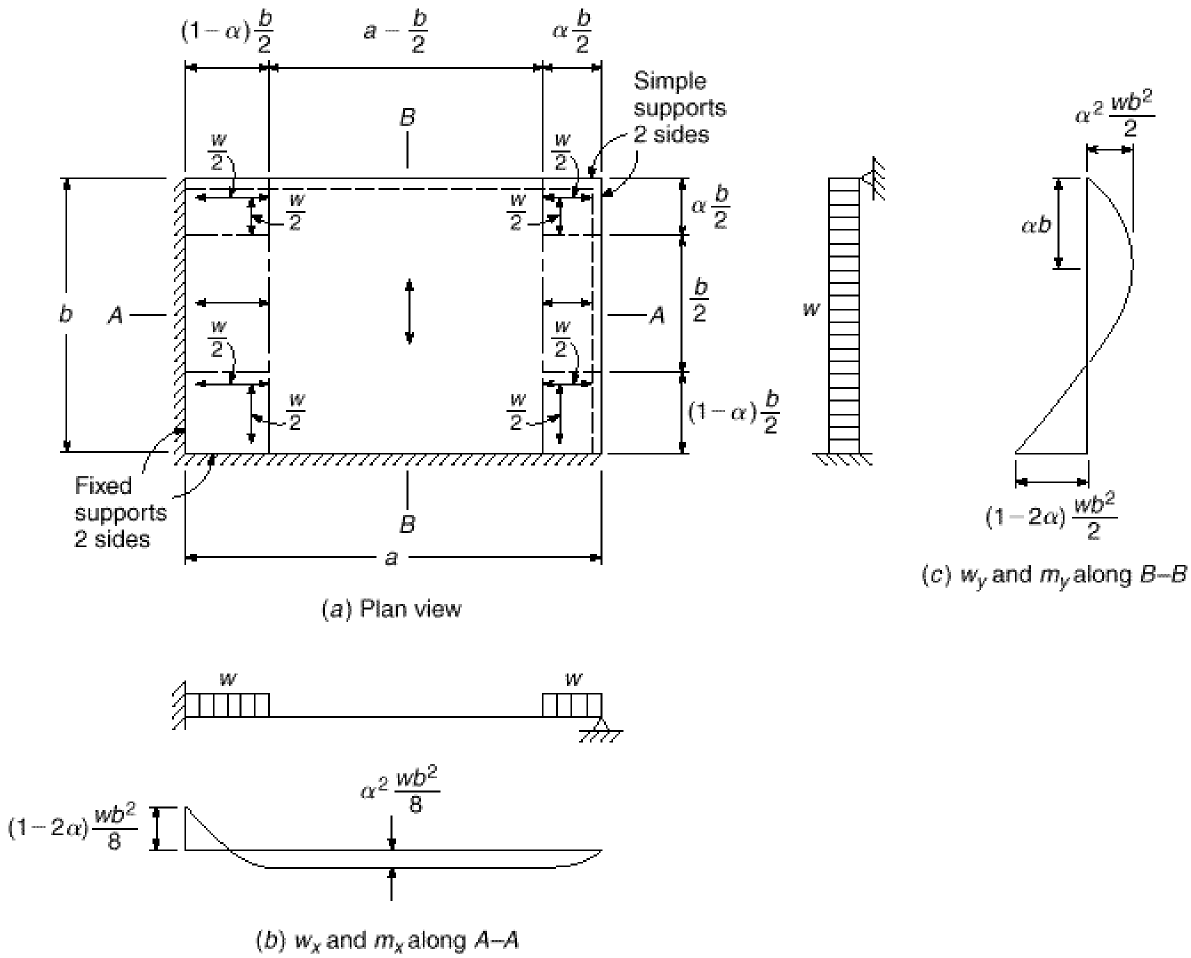


FIGURE 15.7
Rectangular slab with two edges fixed and two edges simply supported.

The cantilever moment at the left support is

$$m_x = (1 - \alpha)\frac{wb}{2}(1 - \alpha)\frac{b}{4} = (1 - \alpha)^2\frac{wb^2}{8} \tag{15.10}$$

and so the negative moment at the left support is

$$m_{xs} = (1 - \alpha)^2\frac{wb^2}{8} - \alpha^2\frac{wb^2}{8} = (1 - 2\alpha)\frac{wb^2}{8} \tag{15.11}$$

For reference, the ratio of negative to positive moments in the *X* direction middle strip is

$$\frac{m_{xs}}{m_{xf}} = \frac{1 - 2\alpha}{\alpha^2} \tag{15.12}$$

The moments in the *X* direction edge strips are one-half of those in the middle strips because the load is half as great.

It is reasonable to choose the same ratio between support and span moments in the *Y* direction as in the *X* direction. Accordingly, the distance from the right support,

Fig. 15.7*c*, to the maximum positive moment section is chosen as · *b*. It follows that the maximum positive moment is

$$m_{yf} = \cdot\, wb \times \frac{\cdot\, b}{2} = \cdot^{2}\, \frac{wb^2}{2} \tag{15.13}$$

Applying the same methods as used for the *X* direction shows that the negative support moment in the *Y* direction middle strips is

$$m_{ys} = \cdot 1 - 2 \cdot\, \cdot\, \frac{wb^2}{2} \tag{15.14}$$

It is easily confirmed that the moments in the *Y* direction edge strips are just one-eighth of those in the *Y* direction middle strip.

With the above expressions, all of the moments in the slab can be found once a suitable value for · is chosen. From Eq. (15.12), it can be confirmed that values of · from 0.35 to 0.39 give corresponding ratios of negative to positive moments from 2.45 to 1.45, the range recommended by Hillerborg. For example, if it is decided that support moments are to be twice the span moments, the value of · should be 0.366, and the negative and positive moments in the central strip in the *Y* direction are, respectively, $0.134wb^2$ and $0.067wb^2$. In the middle strip in the *X* direction, moments are one-fourth those values; and in the edge strips in both directions, they are one-eighth of those values.

EXAMPLE 15.1

Rectangular slab with fixed edges. Figure 15.8 shows a typical interior panel of a slab floor in which support is provided by beams on all column lines. Normally proportioned beams will be stiff enough, both flexurally and torsionally, that the slab can be assumed fully restrained on all sides. Clear spans for the slab, face to face of beams, are 25 ft and 20 ft as shown. The floor must carry a service live load of 150 psf, using concrete with $f'_c = 3000$ psi and steel with $f_y = 60{,}000$ psi. Find the moments at all critical sections, and determine the required slab thickness and reinforcement.

Solution. The minimum slab thickness required by the ACI Code can be found from Eq. (13.8*b*), with $l_n = 25$ ft and · = 1.25:

$$h = \frac{25 \times 12 \cdot 0.8 + 60 \cdot 200 \cdot}{36 + 9 \times 1.25} = 6.98 \text{ in.}$$

A total depth of 6.75 in. will be selected, for which $w_d = 150 \times 6.75 \cdot 12 = 84$ psf. Applying the load factors of 1.2 and 1.6 to dead load and live load, respectively, determines that the total factored load for design is 340 psf. For strip analysis, discontinuity lines will be selected as shown in Fig. 15.8, with edge strips of width $b \cdot 4 = 20 \cdot 4 = 5$ ft. In the corners, the load is divided equally in the two directions; elsewhere, 100 percent of the load is assigned to the direction indicated by the arrows. A ratio of support moment to span moment of 2.0 will be used. Calculation of moments then proceeds as follows:

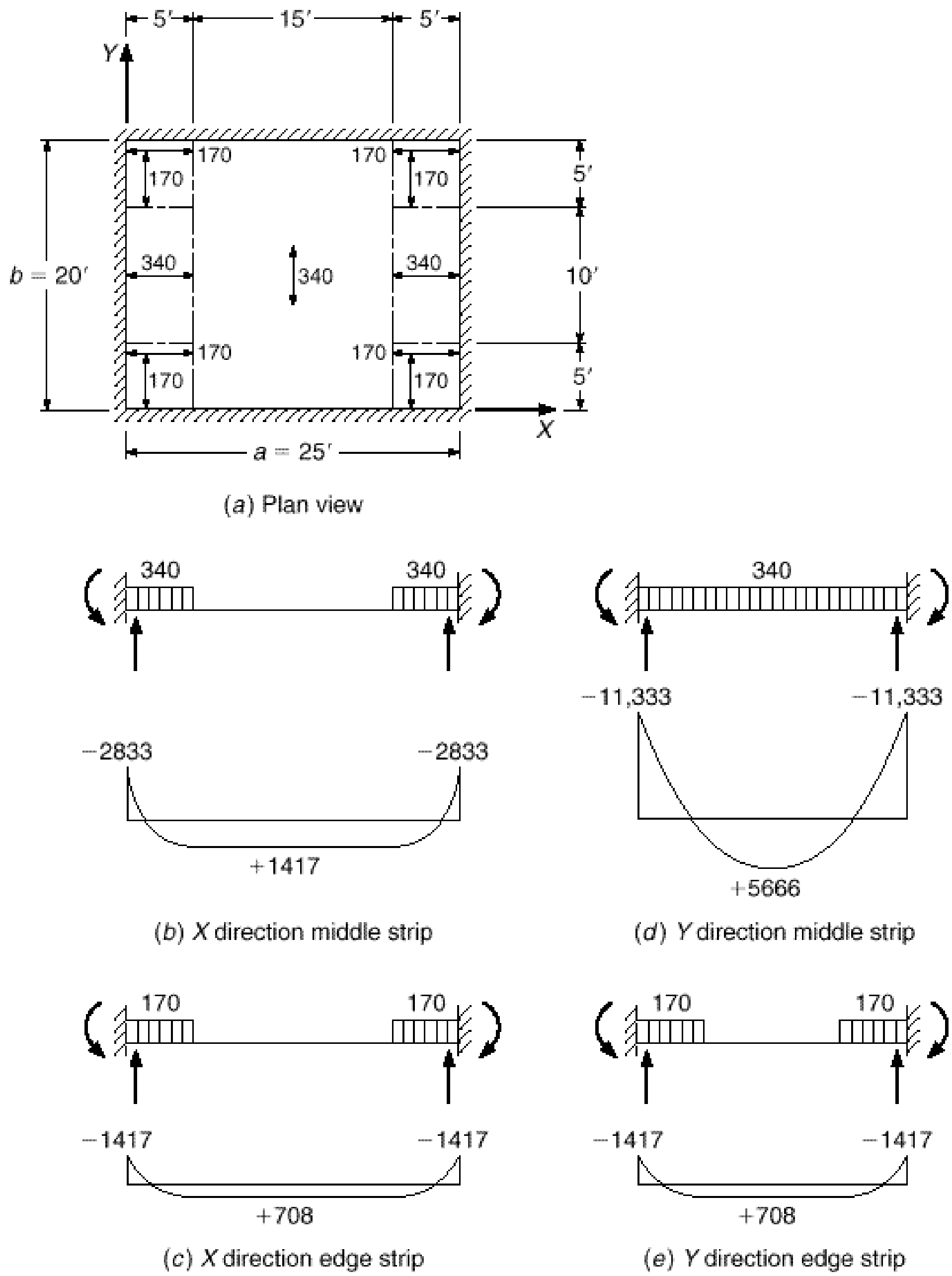
X direction middle strip:

Cantilever: $m_x = \frac{wb^2}{32} = 340 \times \frac{400}{32} = 4250$ ft-lb· ft

Negative: $m_{xs} = 4250 \times \frac{2}{3} = 2833$

Positive: $m_{xf} = 4250 \times \frac{1}{3} = 1417$

FIGURE 15.8
Design example: two-way slab with fixed edges.



X direction edge strips:

Cantilever: $m_x = \dfrac{wb^2}{64} = 340 \times \dfrac{400}{64} = 2125 \text{ ft-lb}\cdot\text{ft}$

Negative: $m_{xs} = 2125 \times \dfrac{2}{3} = 1417$

Positive: $m_{xf} = 2125 \times \dfrac{1}{3} = 708$

Y direction middle strip:

Cantilever: $$m_y = \frac{wb^2}{8} = 340 \times \frac{400}{8} = 17{,}000 \text{ ft-lb}\cdot\text{ft}$$

Negative: $$m_{ys} = 17{,}000 \times \frac{2}{3} = 11{,}333$$

Positive: $$m_{yf} = 17{,}000 \times \frac{1}{3} = 5666$$

Y direction edge strips:

Cantilever: $$m_y = \frac{wb^2}{64} = 340 \times \frac{400}{64} = 2125 \text{ ft-lb}\cdot\text{ft}$$

Negative: $$m_{ys} = 2125 \times \frac{2}{3} = 1417$$

Positive: $$m_{yf} = 2125 \times \frac{1}{3} = 708$$

Strip loads and moment diagrams are as shown in Fig. 15.8. According to ACI Code 7.12, the minimum steel required for shrinkage and temperature crack control is $0.0018 \times 6.75 \times 12 = 0.146$ in^2/ft strip. With a total depth of 6.75 in., with $\frac{3}{4}$ in. concrete cover, and with estimated bar diameters of $\frac{1}{2}$ in., the effective depth of the slab in the short direction will be 5.75 in., and in the long direction, 5.25 in. Accordingly, the flexural reinforcement ratio provided by the minimum steel acting at the smaller effective depth is

$$\rho_{min} = \frac{0.146}{5.25 \times 12} = 0.0023$$

From Table A.5a of Appendix A, $R = 134$, and the flexural design strength is

$$\phi m_n = \phi Rbd^2 = \frac{0.90 \times 134 \times 12 \times 5.25^2}{12} = 3324 \text{ ft-lb}\cdot\text{ft}$$

Comparing this with the required moment resistance shows that the minimum steel will be adequate in the *X* direction in both middle and edge strips and in the *Y* direction edge strips. No. 3 (No. 10) bars at 9 in. spacing will provide the needed area. In the *Y* direction middle strip, for negative bending,

$$R = \frac{m_u}{\phi bd^2} = \frac{11{,}333 \times 12}{0.90 \times 12 \times 5.75^2} = 381$$

and from Table A.5a, the required reinforcement ratio is 0.0069. The required steel is then

$$A_s = 0.0069 \times 12 \times 5.75 = 0.48 \text{ in}^2\cdot\text{ft}$$

This will be provided with No. 5 (No. 16) bars at 8 in. on centers. For positive bending,

$$R = \frac{5666 \times 12}{0.90 \times 12 \times 5.75^2} = 190$$

for which $\rho = 0.0033$, and the required positive steel area per strip is

$$A_s = 0.0033 \times 12 \times 5.75 = 0.23 \text{ in}^2\cdot\text{ft}$$

to be provided by No. 4 (No. 13) bars on 10 in. centers. Note that slight adjustments downward and upward have been made in the steel required at negative and positive bending sections, as permitted by ACI Code 8.4, to arrive at practical bar spacings. Note also that all bar spacings are less than $2h = 2 \times 6.75 = 13.5$ in., as required by the Code, and that the reinforcement ratios are well below the value for a tension-controlled section of 0.0135.

Negative bar cutoff points can easily be calculated from the moment diagrams. For the X direction middle strip, the point of inflection a distance x from the left edge is found as follows:

$$1700x - 2833 - 340\frac{x^2}{2} = 0$$

$$x = 2.11 \text{ ft}$$

According to the Code, the negative bars must be continued at least d or $12d_b$ beyond that point, requiring a 6 in. extension in this case. Thus, the negative bars will be cut off $2.11 + 0.50 = 2.61$ ft, say 2 ft 8 in., from the face of support. The same result is obtained for the X direction edge strips and the Y direction edge strips. For the Y direction middle strip, the distance $y = 4.23$ ft from face of support to inflection point is found in a similar manner. In this case, with No. 5 (No. 10) bars used, the required extension is 7.5 in., giving a total length past the face of supports of $4.23 + 0.63 = 4.86$ ft or 4 ft 11 in. All positive bars will be carried 6 in. into the face of the supporting beams.

15.6 Unsupported Edges

The slabs considered in the preceding sections, together with the supporting beams, could also have been designed by the methods of Chapter 13. The real power of the strip method becomes evident when dealing with nonstandard problems, such as slabs with an unsupported edge, slabs with holes, or slabs with reentrant corners (L-shaped slabs).

For a slab with one edge unsupported, for example, a reasonable basis for analysis by the simple strip method is that a strip along the unsupported edge takes a greater load per unit area than the actual unit load acting, i.e., that the strip along the unsupported edge acts as a support for the strips at right angles. Such strips have been referred to by Wood and Armer as "strong bands" (Ref. 15.8). A strong band is, in effect, an integral beam, usually having the same total depth as the remainder of the slab but containing a concentration of reinforcement. The strip may be made deeper than the rest of the slab to increase its carrying capacity, but this will not usually be necessary.

Figure 15.9*a* shows a rectangular slab carrying a uniformly distributed factored load w per unit area, with fixed edges along three sides and no support along one short side. Discontinuity lines are chosen as shown. The load on a unit middle strip in the X direction, shown in Fig. 15.9*b*, includes the downward load w in the region adjacent to the fixed left edge and the upward reaction kw in the region adjacent to the free edge. Summing moments about the left end, with moments positive clockwise and with the unknown support moment denoted m_{xs}, gives

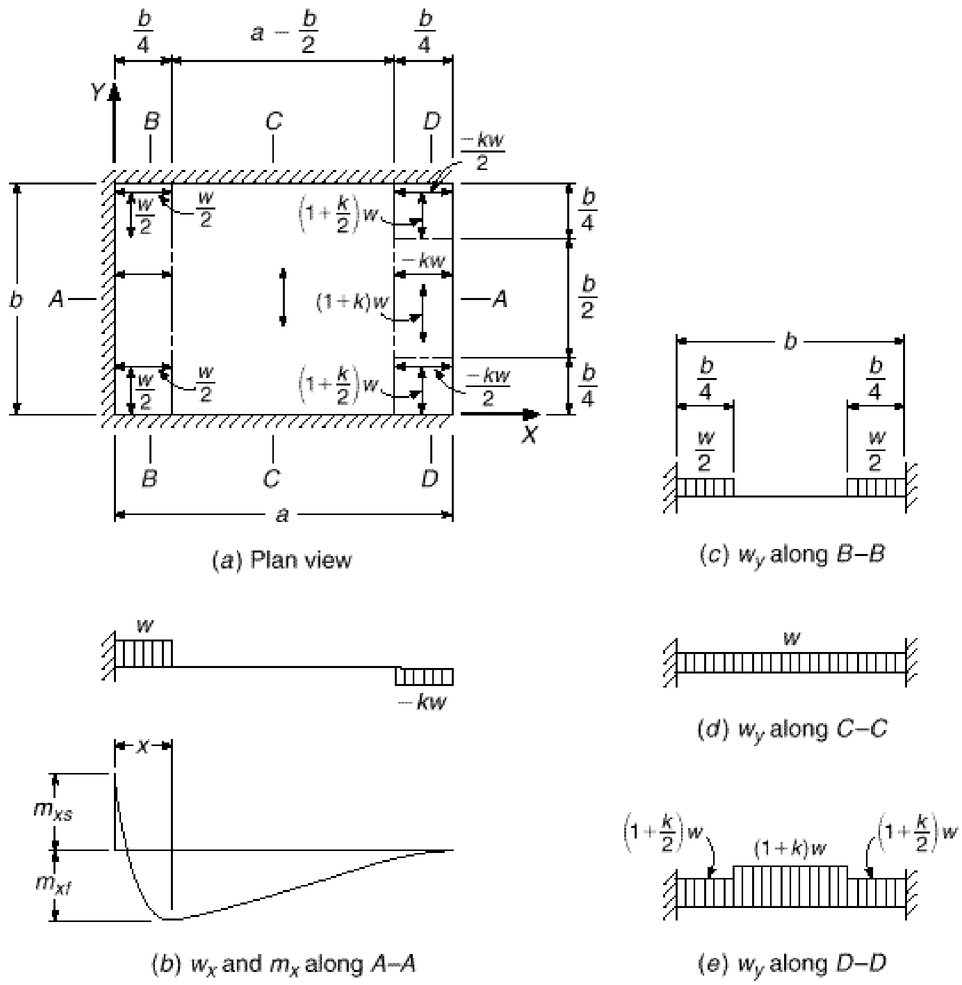
$$m_{xs} + \frac{wb^2}{32} - \frac{kwb}{4}\left(a - \frac{b}{8}\right) = 0$$

from which

$$k = \frac{1 + 32m_{xs}/wb^2}{8(a/b) - 1} \qquad (15.15)$$

Thus, k can be calculated after the support moment is selected.

FIGURE 15.9
Slab with free edge along short side.



The appropriate value of m_{xs} to be used in Eq. (15.15) will depend on the shape of the slab. If a is large relative to b, the strong band in the Y direction at the edge will be relatively stiff, and the moment at the left support in the X direction strips will approach the elastic value for a propped cantilever. If the slab is nearly square, the deflection of the strong band will tend to increase the support moment; a value about half the free cantilever moment might be selected (Ref. 15.14).

Once m_{xs} is selected and k is known, it is easily shown that the maximum span moment occurs when

$$x = (1 - k)\frac{b}{4}$$

It has a value

$$m_{xf} = \frac{kwb^2}{32}\left(\frac{8a}{b} - 3 + k\right) \tag{15.16}$$

The moments in the X direction edge strips are one-half of those in the middle strip. In the Y direction middle strip, Fig. 15.9d, the cantilever moment is $wb^2/8$. Adopting a ratio of support to span moment of 2 results in support and span moments, respectively, of

$$m_{ys} = \frac{wb^2}{12} \tag{15.17a}$$

$$m_{yf} = \frac{wb^2}{24} \tag{15.17b}$$

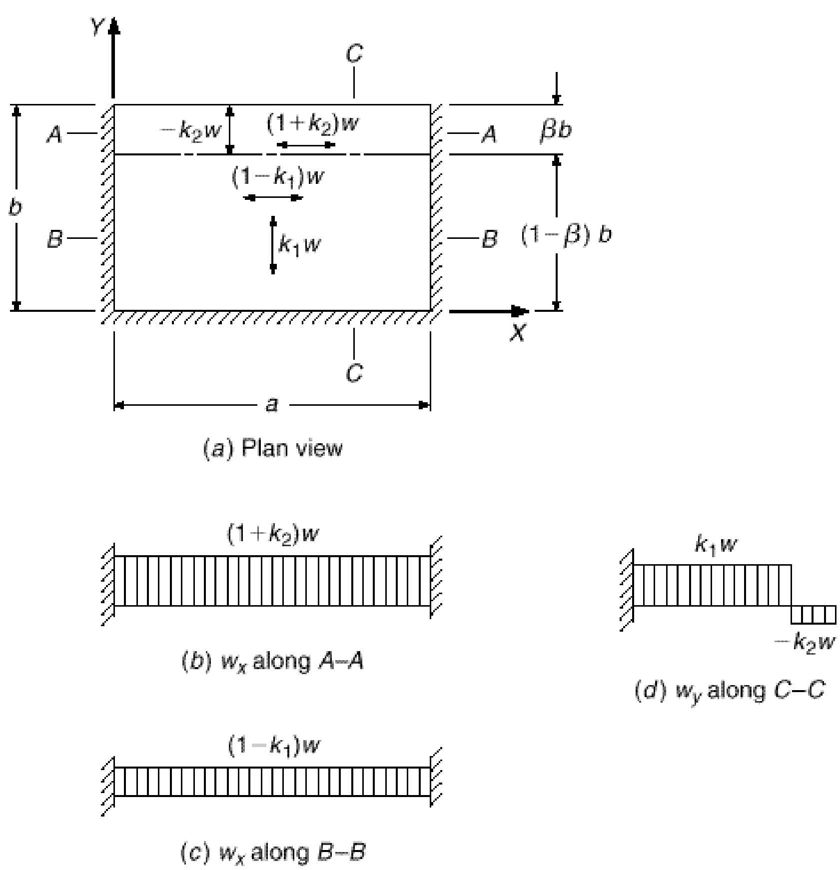
Moments in the Y direction strip adjacent to the fixed edge, Fig. 15.9*c*, will be one-eighth of those values. In the Y direction strip along the free edge, Fig. 15.9*e*, moments can, with slight conservatism, be made equal to $(1 + k)$ times those in the Y direction middle strip.

If the unsupported edge is in the long-span direction, then a significant fraction of the load in the slab central region will be carried in the direction perpendicular to the long edges and the simple distribution shown in Fig. 15.10*a* is more suitable. A strong band along the free edge serves as an integral edge beam, with width · b normally chosen as low as possible considering limitations on tensile reinforcement ratio in the strong band.

For a Y direction strip, with moments positive clockwise,

$$m_{ys} + \frac{1}{2} k_1 w \cdot 1 - \cdot \cdot {}^2 b^2 - k_2 w \cdot b^2 \cdot 1 - \cdot \cdot 2 \cdot = 0$$

FIGURE 15.10
Slab with free edge in long-span direction.



from which

$$k_2 = \frac{k_1(1 - \cdot\,\cdot)^2 + 2m_{ys}/wb^2}{(\cdot\,\cdot)(2 - \cdot\,\cdot)} \tag{15.18}$$

The value of k_1 may be selected so as to make use of the minimum steel in the X direction required by ACI Code 7.12. In choosing m_{ys} to be used in Eq. (15.18) for calculating k_2, one should again recognize that the deflection of the strong band along the free edge will tend to increase the Y direction moment at the supported edge above the propped cantilever value based on zero deflection. A value for m_{ys} of about half the free cantilever moment may be appropriate in typical cases. A high ratio of a/b will permit greater deflection of the free edge through the central region, tending to increase the support moment, and a low ratio will restrict deflection, reducing the support moment.

EXAMPLE 15.2

Rectangular slab with long edge unsupported. The 12 × 19 ft slab shown in Fig. 15.11*a*, with three fixed edges and one long edge unsupported, must carry a uniformly distributed service live load of 125 psf. $f'_c = 4000$ psi, and $f_y = 60{,}000$ psi. Select an appropriate slab thickness, determine all factored moments in the slab, and select reinforcing bars and spacings for the slab.

Solution. The minimum thickness requirements of the ACI Code do not really apply to the type of slab considered here. However, Table 13.5, which controls for beamless flat plates, can be applied conservatively because, although the present slab is beamless along the free edge, it has infinitely stiff supports on the other three edges. From that table, with $l_n = 19$ ft,

$$h = \frac{19 \times 12}{33} = 6.91 \text{ in.}$$

A total depth of 7 in. will be selected. The slab dead load is $150 \times \frac{7}{12} = 88$ psf, and the total factored design load is $1.2 \times 88 + 1.6 \times 125 = 306$ psf.

A strong band 2 ft wide will be provided for support along the free edge. In the main slab, a value $k_1 = 0.45$ will be selected, resulting in a slab load in the Y direction of $0.45 \times 306 = 138$ psf and in the X direction of $0.55 \times 306 = 168$ psf.

First, with regard to the Y direction slab strips, the negative moment at the supported edge will be chosen as one-half the free cantilever value, which in turn will be approximated based on 138 psf over an 11 ft distance from the support face to the center of the strong band. The restraining moment is thus

$$m_{ys} = \frac{1}{2} \times \frac{138 \times 11^2}{2} = 4175 \text{ ft-lb/ft}$$

Then, from Eq. (15.18)

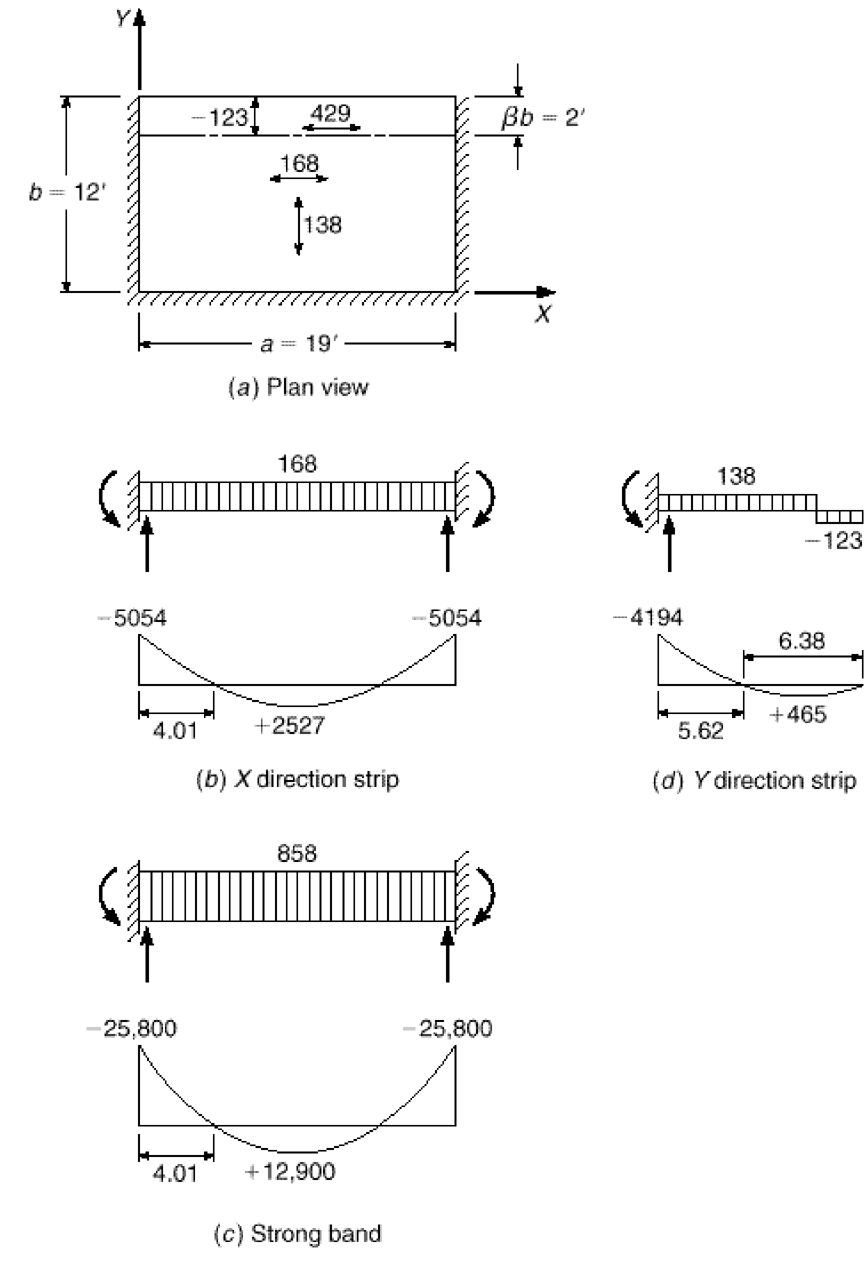
$$k_2 = \frac{0.45(5/6)^2 - 2 \times 4175/(306 \times 144)}{(1/6)(2 - 1/6)} = 0.403$$

Thus, an uplift of $0.403 \times 306 = 123$ psf will be provided for the Y direction strips by the strong band, as shown in Fig. 15.11*d*. For this loading, the negative moment at the left support is

$$m_{ys} = 138 \times \frac{10^2}{2} - 123 \times 2 \times 11 = 4194 \text{ ft-lb/ft}$$

The difference from the original value of 4175 ft-lb/ft is caused by slight rounding errors introduced in the load terms. The statically consistent value of 4194 ft-lb/ft will be used for design. The maximum positive moment in the Y direction strips will be located at the point

FIGURE 15.11
Design example: slab with long edge unsupported.



of zero shear. With y_1 as the distance of that point from the free edge to the zero shear location, and with reference to Fig. 15.11*d*,

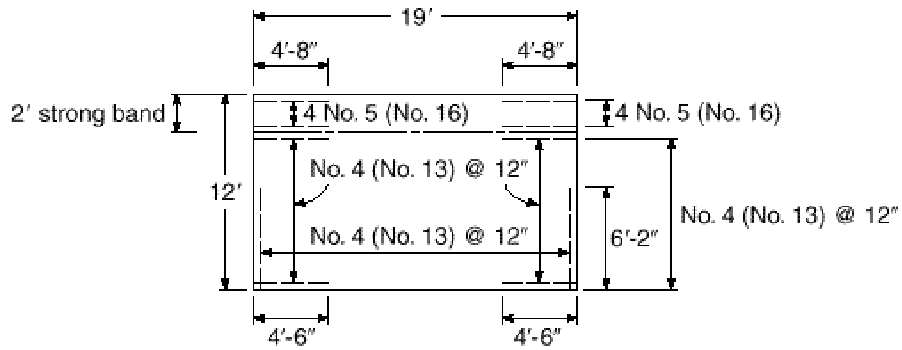
$$123 \times 2 - 138 \cdot y_1 - 2 \cdot = 0$$

from which $y_1 = 3.78$ ft. The maximum positive moment, found at that location, is

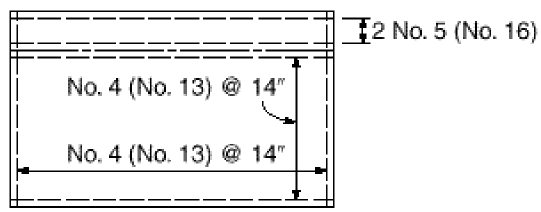
$$m_{yf} = 123 \times 2 \cdot 3.78 - 1 \cdot - 138 \times \frac{1.78^2}{2} = 465 \text{ ft-lb} \cdot \text{ft}$$

For later reference in cutting off bars, the point of inflection is located a distance y_2 from the free edge:

FIGURE 15.11
(*Continued*)



(*e*) Top bars



(*f*) Bottom bars

$$123 \times (2y_2 - 1) - \left(\frac{138}{2}\right)(y_2 - 2)^2 = 0$$

resulting in $y_2 = 6.38$ ft.

For the X direction slab strips, the cantilever moment is

Cantilever: $$m_x = \frac{168 \times 19^2}{8} = 7581 \text{ ft-lb/ft}$$

A ratio of negative to positive moments of 2.0 will be chosen here, resulting in negative and positive moments, respectively, of

Negative: $$m_{xs} = 7581 \times \frac{2}{3} = 5054 \text{ ft-lb/ft}$$

Positive: $$m_{xf} = 7581 \times \frac{1}{3} = 2527 \text{ ft-lb/ft}$$

as shown in Fig. 15.11*b*.

The unit load on the strong band in the X direction is

$$(1 + k_2)w = (1 + 0.403) \times 306 = 429 \text{ psf}$$

so for the 2 ft wide band the load per foot is $2 \times 429 = 858$ psf, as indicated in Fig. 15.11*c*. The cantilever, negative, and positive strong band moments are, respectively,

Cantilever: $$M_x = 858 \times 19^2/8 = 38{,}700 \text{ ft-lb}$$

Negative: $$M_{xs} = 38{,}700 \times \frac{2}{3} = 25{,}800 \text{ ft-lb}$$

Positive: $$M_{xf} = 38{,}700 \times \frac{1}{3} = 12{,}900 \text{ ft-lb}$$

With a negative moment of −25,800 ft-lb and a support reaction of $858 \times \frac{19}{2} = 8151$ lb, the point of inflection in the strong band is found as follows:

$$-25{,}800 + 8151x - \frac{858x^2}{2} = 0$$

giving $x = 4.01$ ft. The inflection point in the X direction slab strips will be at the same location.

In designing the slab steel in the X direction, one notes that the minimum steel required by the ACI Code is $0.0018 \times 7 \times 12 = 0.15$ in^2/ft. The effective slab depth in the X direction, assuming $\frac{1}{2}$ in. diameter bars with $\frac{3}{4}$ in. cover, is $7.0 - 1.0 = 6.0$ in. The corresponding flexural reinforcement ratio in the X direction is · $= 0.15 \cdot (12 \times 6) = 0.0021$. From Table A.5a, $R = 124$, and the design strength is

$$\cdot\, m_n = \cdot\, Rbd^2 = \frac{0.90 \times 124 \times 12 \times 6^2}{12} = 4018 \text{ ft-lb} \cdot \text{ft}$$

It is seen that the minimum slab steel required by the Code will provide for the positive bending moment of 2527 ft-lb/ft. The requirement of 0.15 in^2/ft could be met by No. 3 (No. 10) bars at 9 in. spacing, but to reduce placement costs, No. 4 (No. 13) bars at the maximum permitted spacing of $2h = 14$ in. will be selected, providing 0.17 in^2/ft. The X direction negative moment of 5054 ft-lb/ft requires

$$R = \frac{m_u}{\cdot\, bd^2} = \frac{5054 \times 12}{0.90 \times 12 \times 6^2} = 156$$

and Table A.5a indicates that the required · $= 0.0027$. Thus, the negative bar requirement is $A_s = 0.0027 \times 12 \times 6 = 0.19$ in^2/ft. This will be provided by No. 4 (No. 13) bars at 12 in. spacing, continued $4.01 \times 12 + 6 = 54$ in., or 4 ft 6 in., from the support face.

In the Y direction, the effective depth will be one bar diameter less than in the X direction, or 5.5 in. Thus, the flexural reinforcement ratio provided by the shrinkage and temperature steel is · $= 0.15 \cdot (12 \times 5.5) = 0.0023$. This results in $R = 135$, so the design strength is

$$\cdot\, m_n = \frac{0.90 \times 135 \times 12 \times 5.5^2}{12} = 3675 \text{ ft-lb} \cdot \text{ft}$$

well above the requirement for positive bending of 473 ft-lb/ft. No. 4 (No. 13) bars at 14 in. will be satisfactory for positive steel in this direction also. For the negative moment of 4194 ft-lb/ft,

$$R = \frac{4194 \times 12}{0.90 \times 12 \times 5.5^2} = 154$$

and from Table A.5a, the required · $= 0.0027$. The corresponding steel requirement is $0.0027 \times 12 \times 5.5 = 0.18$ in^2/ft. No. 4 (No. 13) bars at 12 in. will be used, and they will be extended $5.62 \times 12 + 6 = 74$ in., or 6 ft 2 in., past the support face.

In the strong band, the positive moment of 13,100 ft-lb requires

$$R = \frac{12{,}900 \times 12}{0.90 \times 24 \times 6^2} = 199$$

The corresponding reinforcement ratio is 0.0034, and the required bar area is $0.0034 \times 24 \times 6 = 0.49$ in^2. This can be provided by two No. 5 (No. 16) bars. For the negative moment of 26,200 ft-lb,

$$R = \frac{25{,}800 \times 12}{0.90 \times 24 \times 6^2} = 398$$

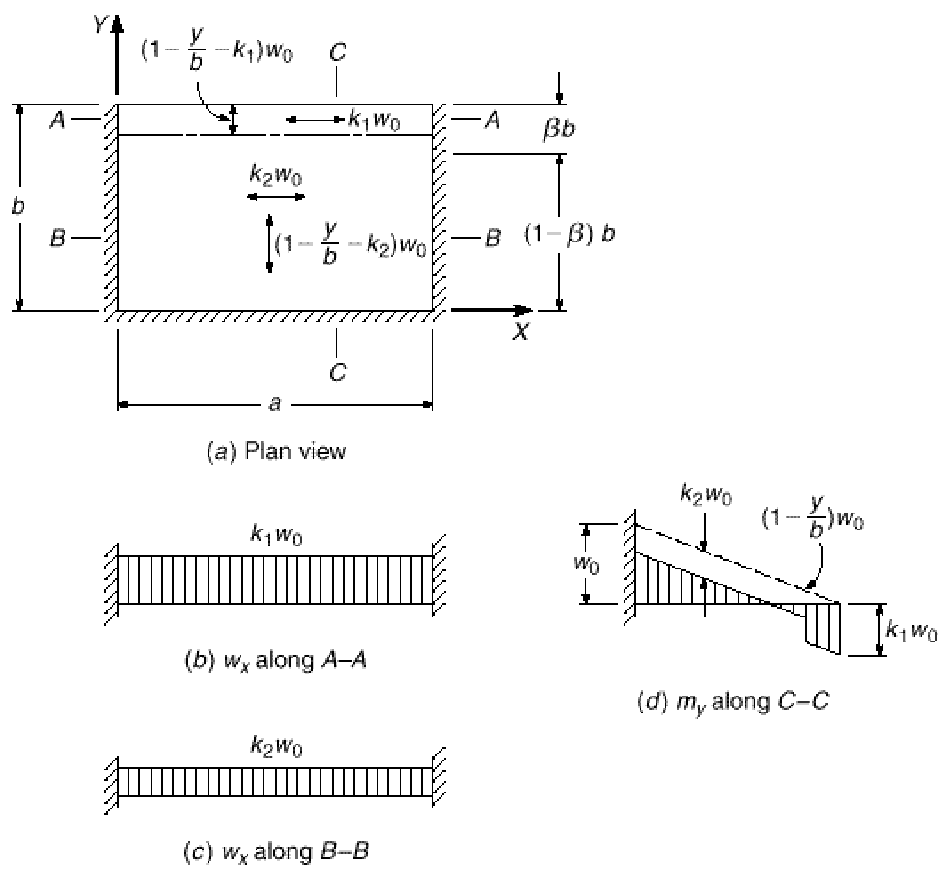
resulting in · $= 0.0070$, and required steel equal to $0.0070 \times 24 \times 6 = 1.01$ in^2. Four No. 5 (No. 16) bars, providing an area of 1.23 in^2, will be used, and they will be cut off $4.01 \times 12 + 7.5 = 56$ in., or 4 ft 8 in., from the support face.

The final arrangement of bar reinforcement is shown in Fig. 15.11*e* and *f*. Negative bar cutoff locations are as indicated, and development by embedded lengths into the supports will be provided. All positive bars in the slab and strong band will be carried 6 in. into the support faces.

A design problem commonly met in practice is that of a slab supported along three edges and unsupported along the fourth, with a distributed load that increases linearly from zero along the free edge to a maximum at the opposite supported edge. Examples include the wall of a rectangular tank subjected to liquid pressure and earth-retaining walls with buttresses or counterforts (see Section 17.1).

Figure 15.12 shows such a slab, with load of intensity w_0 at the long, supported edge, reducing to zero at the free edge. In the main part of the slab, a constant load k_2w_0 is carried in the X direction, as shown in Fig. 15.12*c*; thus, a constant load k_2w_0 is deducted from the linear varying load in the Y direction, as shown in Fig. 15.12*d*. Along the free edge, a strong band of width βb is provided, carrying a load k_1w_0, as in Fig. 15.12*a*, and so providing an uplift load equal to that amount at the end of the Y direction strip in Fig. 15.12*d*. The choice of k_1 and k_2 depends on the ratio of a/b. If this ratio is high, k_2 should be chosen with regard to the minimum slab reinforcement required by the ACI Code. The value of k_1 is then calculated by statics, based on a

FIGURE 15.12
Slab with one free edge and linearly varying load.



selected value of the restraining moment at the fixed edge, say one-half of the free cantilever value. In many cases it will be convenient to let k_1 equal k_2. Then it is the support moment that follows from statics. The value of · is selected as low as possible considering the upper limit on tensile reinforcement ratio in the strong band imposed by the Code for beams. The strong band is designed for a load of intensity $k_1 w_0$ distributed uniformly over its width · b.

15.7 Slabs with Holes

Slabs with small openings can usually be designed as if there were no openings, replacing the interrupted steel with bands of reinforcing bars of equivalent area on either side of the opening in each direction (see Section 13.12). Slabs with larger openings must be treated more rigorously. The strip method offers a rational and safe basis for design in such cases. Integral load-carrying beams are provided along the edges of the opening, usually having the same depth as the remainder of the slab but with extra reinforcement, to pick up the load from the affected regions and transmit it to the supports. In general, these integral beams should be chosen so as to carry the loads most directly to the supported edges of the slab. The width of the strong bands should be selected so that the reinforcement ratios · are at or below the value required to produce a tension-controlled member (i.e., $\cdot_t \geq 0.005$ and · $= 0.90$). Doing so will ensure ductile behavior of the slab.

Use of the strip method for analysis and design of a slab with a large central hole will be illustrated by the following example.

EXAMPLE 15.3

Rectangular slab with central opening. Figure 15.13*a* shows a 16 × 28 ft slab with fixed supports along all four sides. A central opening 4 × 8 ft must be accommodated. Estimated slab thickness, from Eq. (13.8*b*), is 7 in. The slab is to carry a uniformly distributed factored load of 300 psf, including self-weight. Devise an appropriate system of strong bands to reinforce the opening, and determine moments to be resisted at all critical sections of the slab.

Solution. The basic pattern of discontinuity lines and load dispersion will be selected according to Fig. 15.5. Edge strips are defined having width $\frac{16}{4} = 4$ ft. In the corners, the load is equally divided in the two directions. In the central region, 100 percent of the load is assigned to the Y direction, while along the central part of the short edges, 100 percent of the load is carried in the X direction. Moments for this "basic case" without the hole will be calculated and later used as a guide in selecting moments for the actual slab with hole. A ratio of support to span moments of 2.0 will be used generally, as for the previous examples. Moments for the slab, neglecting the hole, would then be as follows:

X direction middle strips:

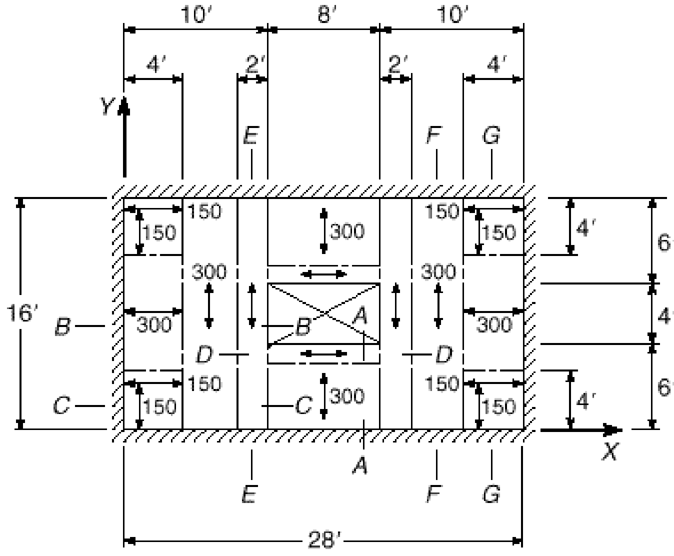
Cantilever: $$m_x = \frac{wb^2}{32} = 300 \times \frac{16^2}{32} = 2400 \text{ ft-lb} \cdot \text{ft}$$

Negative: $$m_{xs} = 2400 \times \frac{2}{3} = 1600$$

Positive: $$m_{xf} = 2400 \times \frac{1}{3} = 800$$

X direction edge strips are $\frac{1}{2}$ middle strip values.

FIGURE 15.13
Design example: slab with central hole.



(*a*) Plan view

Y direction middle strips:

Cantilever: $m_y = \dfrac{wb^2}{8} = 300 \times \dfrac{16^2}{8} = 9600$ ft-lb·ft

Negative: $m_{ys} = 9600 \times \dfrac{2}{3} = 6400$

Positive: $m_{yf} = 9600 \times \dfrac{1}{3} = 3200$

Y direction edge strips are $\frac{1}{8}$ middle strip values.

Because of the hole, certain strips lack support at one end. To support them, 1 ft wide strong bands will be provided in the *X* direction at the long edges of the hole and 2 ft wide strong bands in the *Y* direction on each side of the hole. The *Y* direction bands will provide for the reactions of the *X* direction bands. With the distribution of loads shown in Fig. 15.13*a*, strip reactions and moments are found as follows:

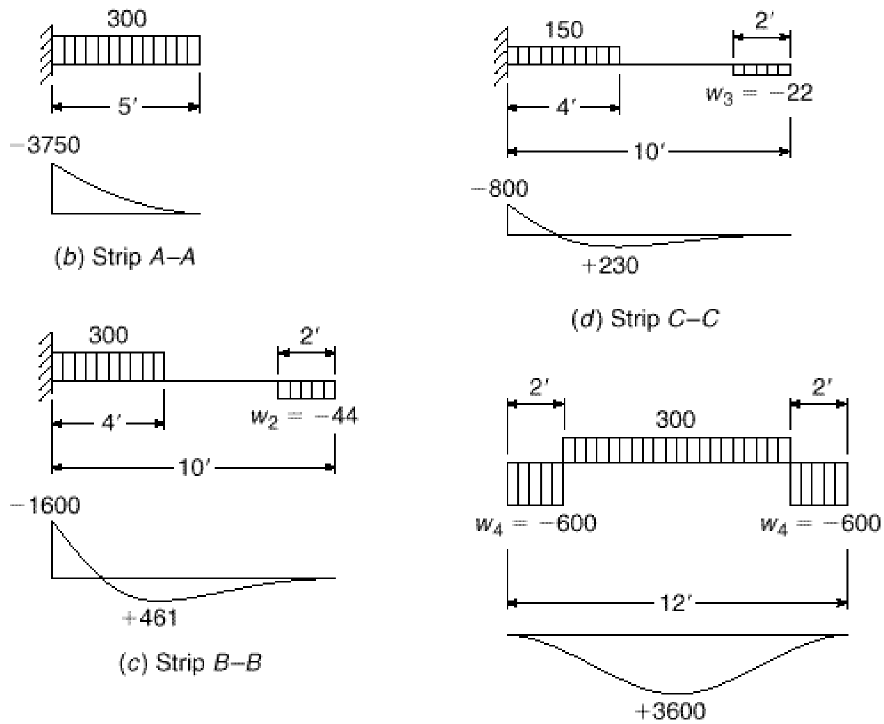
Strip A-A
It may at first be assumed that propped cantilever action is obtained, with the restraint moment along the slab edge taken as 6400 ft-lb/ft, the same as for the basic case. Summing moments about the left end of the loaded strip then results in

$$w_1 = \frac{300 \times 6 \times 3 - 6400}{1 \times 5.5} = -182 \text{ psf}$$

The negative value indicates that the cantilever strips are serving as supports for strip *D-D*, and in turn for the strong bands in the *Y* direction, which is hardly a reasonable assumption. Instead, a discontinuity line will be assumed 5 ft from the support, as shown in Fig. 15.13*b*, terminating the cantilever and leaving the 1 ft strip *D-D* along the edge of the opening in the *X* direction to carry its own load. It follows that the support moment in the cantilever strip is

Negative: $m_{ys} = 300 \times 5 \times 2.5 = 3750$ ft-lb·ft

FIGURE 15.13
(*Continued*)



Strip B-B
The restraint moment at the supported edge will be taken to be the same as the basic case, i.e., 1600 ft-lb/ft. Summing moments about the left end of the strip of Fig. 15.13*c* then results in an uplift reaction at the right end, to be provided by strip *E-E*, of

$$w_2 = \frac{300 \times 4 \times 2 - 1600}{2 \times 9} = 44 \text{ psf}$$

The left reaction is easily found to be 1112 lb, and the point of zero shear is 3.70 ft from the left support. The maximum positive moment, at that point, is

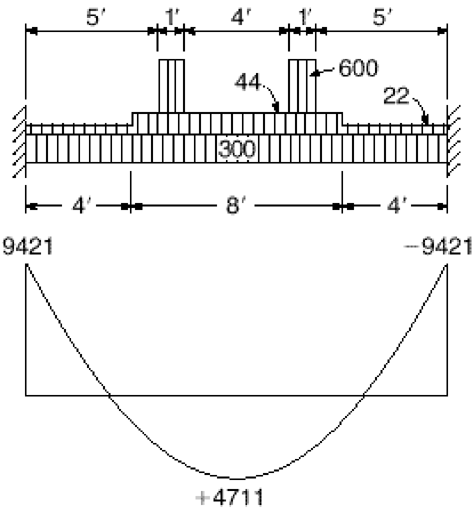
$$\text{Positive:} \qquad m_{xf} = 1112 \times 3.70 - 1600 - 300\,\frac{3.70^2}{2} = 461 \text{ ft-lb/ft}$$

Strip C-C
Negative and positive moments and the reaction to be provided by strip *E-E*, as shown in Fig. 15.13*d*, are all one-half the corresponding values for strip *B-B*.

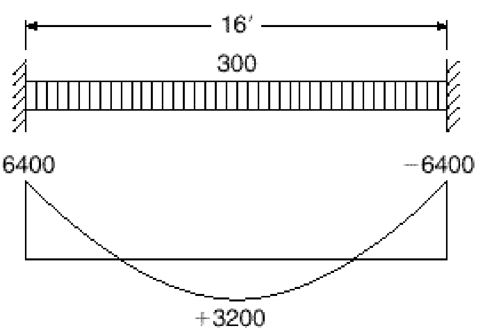
Strip D-D
The 1 ft wide strip carries 300 psf in the *X* direction with reactions provided by the strong bands *E-E*, as shown in Fig. 15.13*e*. The maximum positive moment is

$$m_{xf} = 600 \times 2 \times 5 - 300 \times 4 \times 2 = 3600 \text{ ft-lb/ft}$$

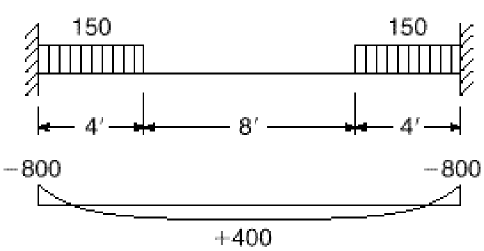
FIGURE 15.13
(*Continued*)



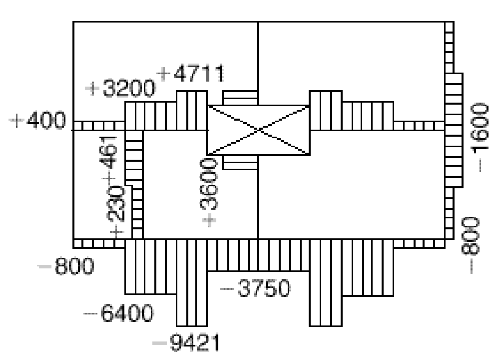
(*f*) Strip *E–E*



(*g*) Strip *F–F*



(*h*) Strip *G–G*



(*i*) Factored moments – symmetrical about both axes

Strip E-E

In reference to Fig. 15.13*f*, the strong bands in the *Y* direction carry the directly applied load of 300 psf plus the 44 psf load from strip *B-B*, the 22 psf load from strip *C-C*, and the 600 psf end reaction from strip *D-D*. For strip *E-E* the cantilever, negative, and positive moments are

Cantilever: $m_y = 300 \times 8 \times 4 + 22 \times 4 \times 2 + 44 \times 4 \times 6 + 600 \times 1 \times 5.5$

$= 14{,}132 \text{ ft-lb}\cdot\text{ft}$

Negative: $m_{ys} = 14{,}132 \times \dfrac{2}{3} = 9421$

Positive: $m_{yf} = 14{,}132 \times \dfrac{1}{3} = 4711$

It should be emphasized that the loads shown are psf and would be multiplied by 2 to obtain loads per foot acting on the strong bands. Correspondingly, the moments just obtained are per foot width and must be multiplied by 2 to give the support and span moments for the 2 ft wide strong band.

Strip F-F

The moments for the *Y* direction middle strip of the basic case may be used without change; thus, in Fig. 15.13*g*,

Negative: $m_{ys} = 6400 \text{ ft-lb}\cdot\text{ft}$

Positive: $m_{yf} = 3200$

Strip G-G

Moments for the *Y* direction edge strips of the basic case are used without change, resulting in

Negative: $m_{ys} = 800 \text{ ft-lb}\cdot\text{ft}$

Positive: $m_{yf} = 400$

as shown in Fig. 15.13*h*.

The final distribution of moments across the negative and positive critical sections of the slab is shown in Fig. 15.13*i*. The selection of reinforcing bars and determination of cutoff points would follow the same methods as presented in Examples 15.1 and 15.2 and will not be given here. Reinforcing bar ratios needed in the strong bands are well below the maximum permitted for the 7 in. slab depth.

It should be noted that strips *B-B*, *C-C*, and *D-D* have been designed as if they were simply supported at the strong band *E-E*. To avoid undesirably wide cracks where these strips pass over the strong band, nominal negative reinforcement should be added in this region. Positive bars should be extended fully into the strong bands.

15.8 Advanced Strip Method

The simple strip method described in the earlier sections of this chapter is not directly suitable for the design of slabs supported by columns (e.g., flat plates) or slabs supported at reentrant corners.[†] For such cases, Hillerborg introduced the advanced strip method (Refs. 15.2, 15.5, 15.12, and 15.13).

[†] However, Wood and Armer, in Ref. 15.8, suggest that beamless slabs with column supports can be solved by the simple strip method through the use of strong bands between columns or between columns and exterior walls.

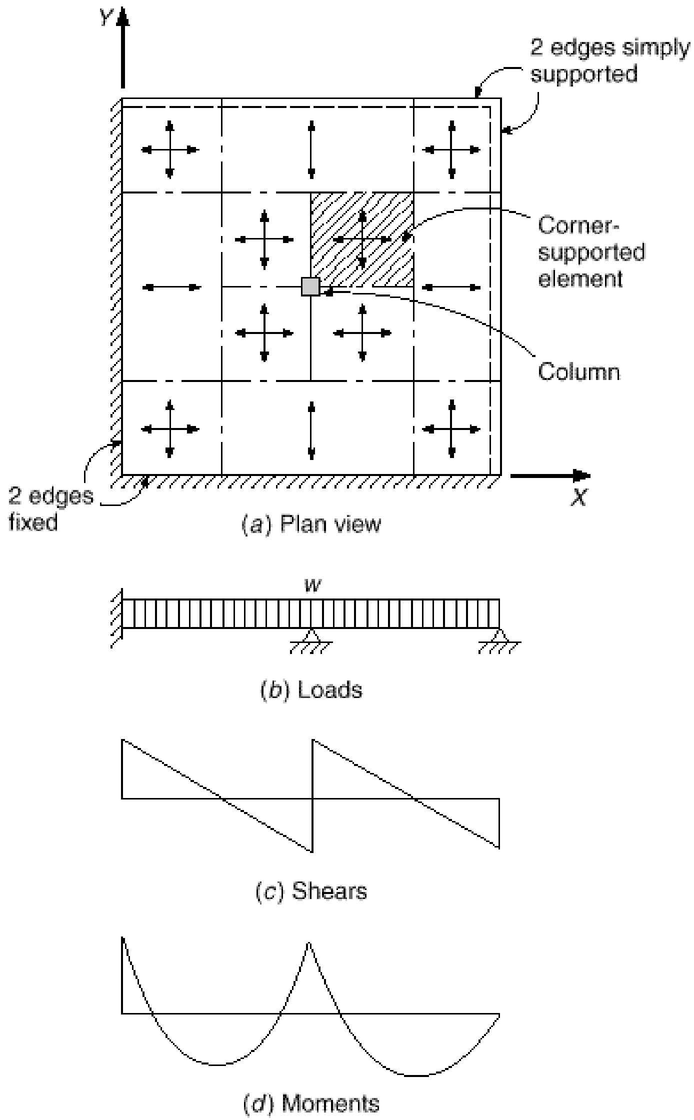


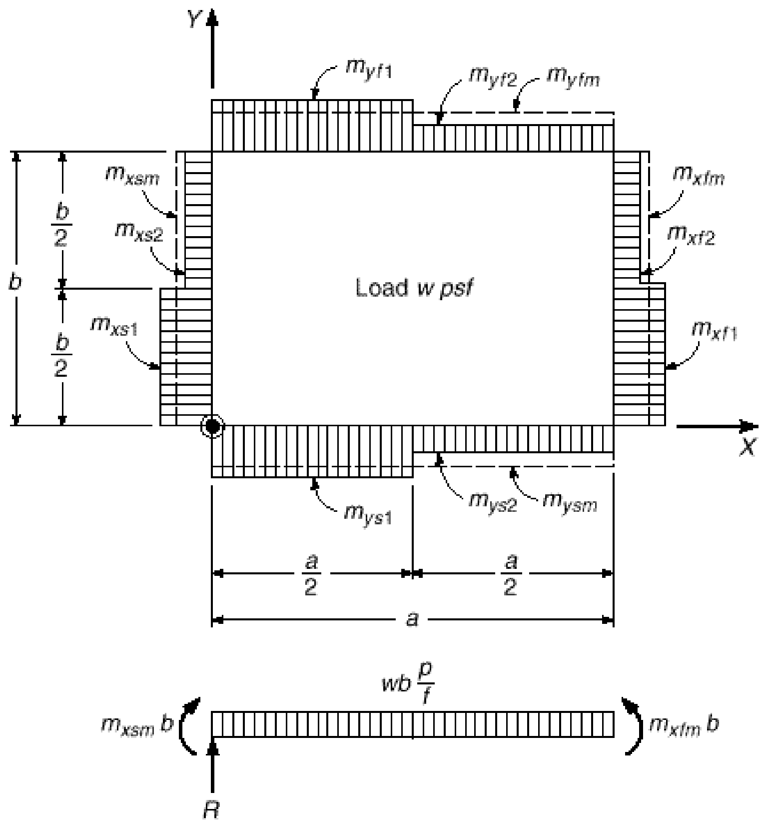
FIGURE 15.14
Slab with central supporting column.

Fundamental to the advanced strip method is the corner-supported element, such as that shown shaded in Fig. 15.14*a*. The corner-supported element is a rectangular region of the slab with the following properties:

1. The edges are parallel to the reinforcement directions.
2. It carries a uniform load *w* per unit area.
3. It is supported at only one corner.
4. No shear forces act along the edges.
5. No twisting moments act along the edges.
6. All bending moments acting along an edge have the same sign or are zero.
7. The bending moments along the edges are the factored moments used to design the reinforcing bars.

A uniformly loaded strip in the *X* direction, shown in Fig. 15.14*b*, will thus have shear and moment diagrams as shown in Fig. 15.14*c* and *d*, respectively. Maximum

FIGURE 15.15
Corner-supported element.



moments are located at the lines of zero shear. The outer edges of the corner-supported element are defined at the lines of zero shear in both the X and Y directions.

A typical corner-supported element, with an assumed distribution of moments along the edges, is shown in Fig. 15.15. It will be assumed that the bending moment is constant along each half of each edge. The vertical reaction is found by summing vertical forces:

$$R = wab \tag{15.19}$$

and moment equilibrium about the Y axis gives

$$m_{xfm} - m_{xsm} = \frac{wa^2}{2} \tag{15.20}$$

where m_{xfm} and m_{xsm} are the mean span and support moments per unit width, and beam sign convention is followed. Similarly,

$$m_{yfm} - m_{ysm} = \frac{wb^2}{2} \tag{15.21}$$

The last two equations are identical with the condition for a corresponding part of a simple strip—Eq. (15.20) spanning in the X direction and Eq. (15.21) in the Y direction—supported at the axis and carrying the load wb or wa per foot. So if the corner-supported element forms a part of a strip, that part should carry 100 percent of the load w in each direction. (This requirement was discussed earlier in Chapter 13 and is simply a requirement of static equilibrium.)

The distribution of moments within the boundaries of a corner-supported element is complex. With the load on the element carried by a single vertical reaction at one corner, strong twisting moments must be present within the element; this contrasts with the assumptions of the simple strip method used previously.

The moment field within a corner-supported element and its edge moments have been explored in great detail in Ref. 15.12. It is essential that the edge moments, given in Fig. 15.15, are used to design the reinforcing bars (i.e., that nowhere within the element will a bar be subjected to a greater moment than at the edges). To meet this requirement, a limitation must be put on the moment distribution along the edges. Based on his studies (Ref. 15.12), Hillerborg has recommended the following restriction on edge moments:

$$m_{xf2} - m_{xs2} = \cdot \frac{wa^2}{2} \qquad (15.22a)$$

with

$$0.25 \le \cdot \le 0.7 \qquad (15.22b)$$

where m_{xf2} and m_{xs2} are the positive and negative X direction moments, respectively, in the outer half of the element, as shown in Fig. 15.15. The corresponding restriction applies in the Y direction. He notes further that for most practical applications, the edge moment distribution shown in Fig. 15.16 is appropriate, with

$$m_{xf1} = m_{xf2} = m_{xfm} \qquad (15.23)$$

$$m_{xs2} = 0 \qquad (15.24a)$$

$$m_{xs1} = 2m_{xsm} \qquad (15.24b)$$

(Alternatively, it is suggested in Ref. 15.14 that negative support moments across the column line be taken at $1.5m_{xsm}$ in the half-element width by the column and at

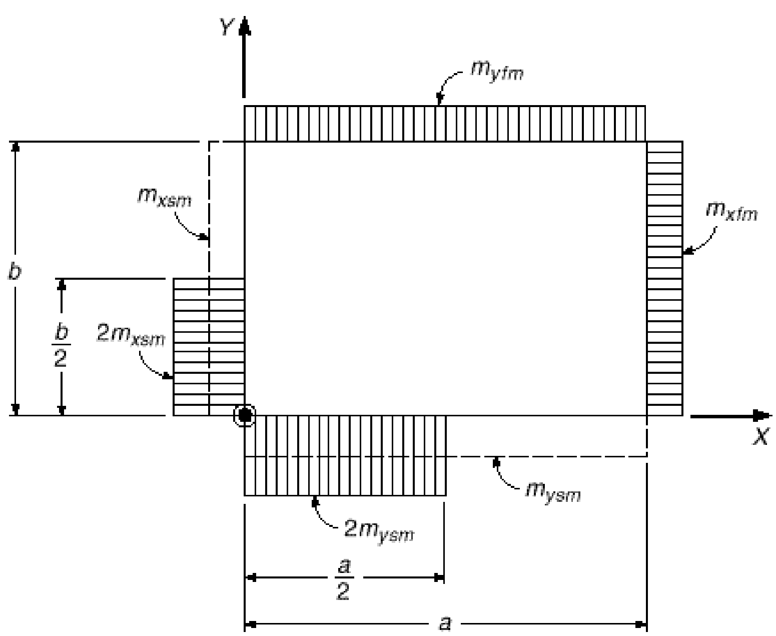


FIGURE 15.16
Recommended distribution of moments for typical corner-supported element.

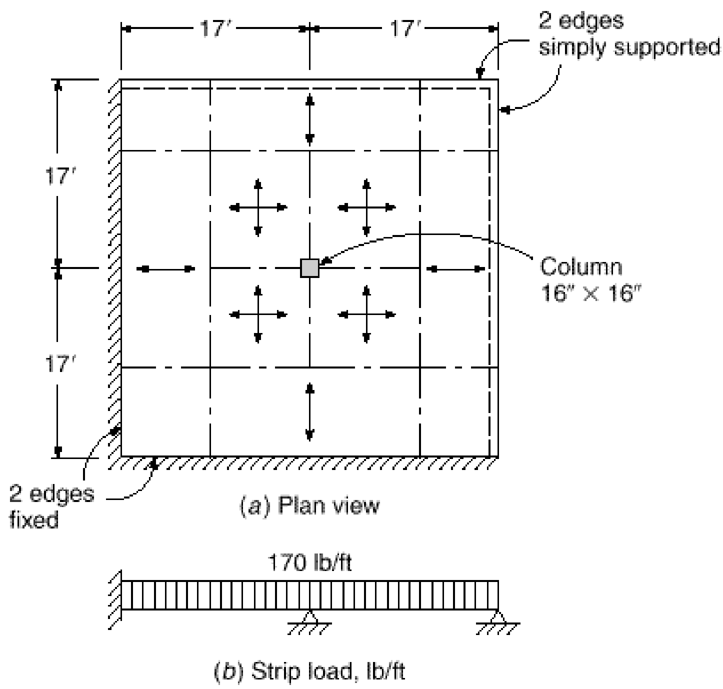
$0.5m_{xsm}$ in the remaining outside half-element width.) Positive reinforcement in the span should be carried through the whole corner-supported element. The negative reinforcement corresponding to $m_{xs1} - m_{xs2}$ in Fig. 15.15 must be extended at least $0.6a$ from the support. The remaining negative steel, if any, should be carried through the whole corner-supported element. The corresponding restrictions apply in the Y direction.

In practical applications, corner-supported elements are combined with each other and with parts of one-way strips, as shown in Fig. 15.14, to form a system of strips. In this system, each strip carries the total load w, as discussed earlier. In laying out the elements and strips, the concentrated corner support for the element may be assumed to be at the center of the supporting column, as shown in Fig. 15.14, unless supports are of significant size. In that case, the corner support may be taken at the corner of the column, and an ordinary simple strip may be included that spans between the column faces, along the edge of the corner-supported elements. Note in the figure that the corner regions of the slab are not included in the main strips that include the corner-supported elements. These may safely be designed for one-third of the corresponding moments in the main strips (Ref. 15.13).

EXAMPLE 15.4

Edge-supported flat plate with central column. Figure 15.17a illustrates a flat plate with overall dimensions 34 × 34 ft, with fixed supports along the left and lower edges in the sketch, hinged supports at the right and upper edges, and a single central column 16 in. square. It must carry a service live load of 40 psf over its entire surface plus its own weight and an additional superimposed dead load of 7 psf. Find the moments at all critical sections, and determine the required slab thickness and reinforcement. Material strengths are specified at f_y = 60,000 psi and f'_c = 4000 psi.

FIGURE 15.17
Design example: edge-supported flat plate with central column.



Solution. A trial slab depth will be chosen based on Table 13.5, which governs for flat plates. It will be conservative for the present case, where continuous support is provided along the outer edges.

$$h = \frac{17 \times 12}{33} = 6.18 \text{ in.}$$

A thickness of 6.5 in. will be selected tentatively, for which the self-weight is $150 \times 6.5 \cdot 12 = 81$ psf. The total factored load to be carried is thus:

$$w_u = 1.2 \cdot 81 + 7 \cdot + 1.6 \times 40 = 170 \text{ psf}$$

The average strip moments in the *X* direction in the central region caused by the load of 170 psf are found by elastic theory and are shown in Fig. 15.17*c*. The analysis in the *Y* direction is identical. The points of zero shear (and maximum moments) are located 9.11 ft to the left of the column, and 10.32 ft to the right, as indicated. These dimensions determine the size of the four corner-supported elements.

Moments in the slab are then determined according to the preceding recommendations. At the fixed edge along the left side of the main strips, the moment m_{xs} is simply the moment per foot strip from the elastic analysis, 3509 ft-lb/ft. At the left edge of the corner-supported element in the left span,

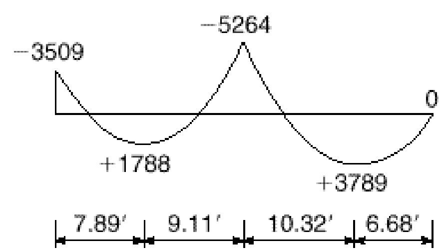
$$m_{xf1} = m_{xf2} = m_{xfm} = 1788 \text{ ft-lb} \cdot \text{ft}$$

Along the centerline of the slab, over the column, following the recommendations shown in Fig. 15.16,

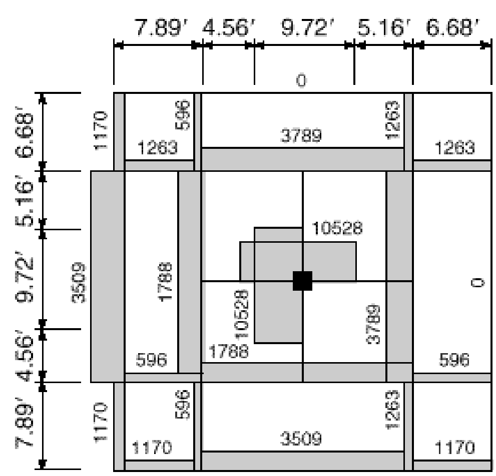
$$m_{xs2} = 0$$

$$m_{xs1} = 2m_{xsm} = 10{,}528 \text{ ft-lb} \cdot \text{ft}$$

FIGURE 15.17
(*Continued*)

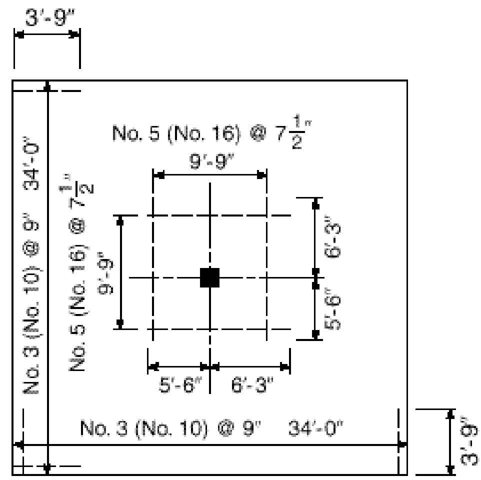


(c) Moments m_x, ft-lb/ft

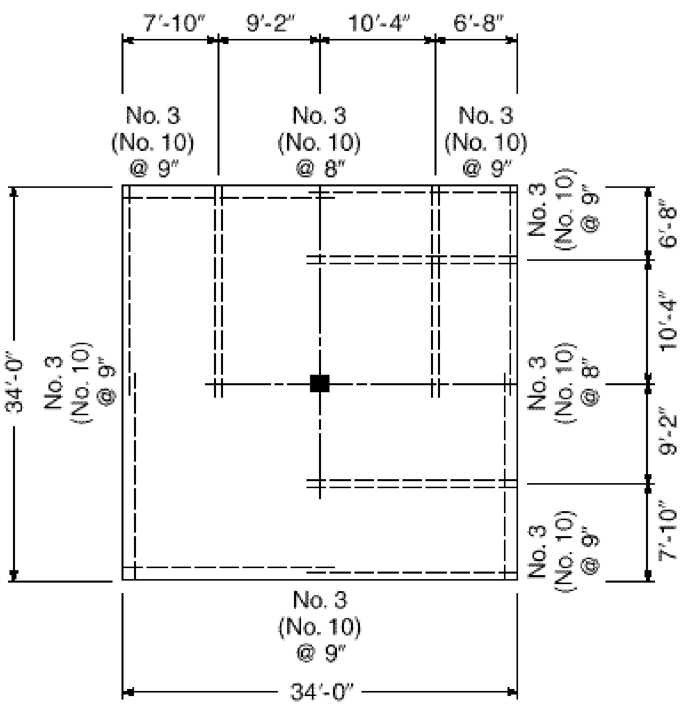


(d) Factored moments per 1 foot strip

FIGURE 15.17
(*Continued*)



(*e*) Top steel



(*f*) Bottom steel

At the right edge of the corner-supported element in the right span,

$$m_{xf1} = m_{xf2} = m_{xfm} = 3789 \text{ ft-lb}\cdot\text{ft}$$

At the outer, hinge-supported edge, all moments are zero. Make a check of the · values, using Eq. (15.22*b*), and note from Eq. (15.20) that $wa^2 \cdot 2 = m_{xfm} - m_{xsm}$. Thus, in the left span,

$$\cdot = \frac{m_{xf2} - m_{xs2}}{wa^2 \cdot 2} = \frac{1788 - 0}{1788 + 5264} = 0.25$$

and in the right span,

$$\cdot = \frac{3789 - 0}{3789 + 5264} = 0.42$$

Because both values are within the range of 0.25 to 0.75, the proposed distribution of moments is satisfactory. If the first value had been below the lower limit of 0.25, the negative moment in the column half-strip might have been reduced from 10,528 ft-lb per ft, and the negative moment in the adjacent half-strip might have been increased above the 0 value used. Alternatively, the total negative moment over the column might have been somewhat decreased, with a corresponding increase in span moments.

Moments in the *Y* direction correspond throughout, and all results are summarized in Fig. 15.17*d*. Moments in the strips adjacent to the supported edges are set equal to one-third of those in the adjacent main strips.

With moments per ft strip known at all critical sections, the required reinforcement is easily found. With a $\frac{3}{4}$ in. concrete cover and $\frac{1}{2}$ in. bar diameter, in general the effective depth of the slab will be 5.5 in. Where bar stacking occurs—i.e., over the central column and near the intersection of the two fixed edges—an average effective depth equal to 5.25 in. will be used. This will result in reinforcement identical in the two directions and will simplify construction.

For the 6.5 in. thick slab, minimum steel for shrinkage and temperature crack control is $0.0018 \times 6.5 \times 12 = 0.140$ in^2/ft strip, which will be provided by No. 3 (No. 10) bars at 9 in. spacing. The corresponding flexural reinforcement ratio is

$$\cdot_{min} = \frac{0.140}{5.5 \times 12} = 0.0021$$

Interpolating from Table A.5a of Appendix A makes $R = 124$, and the design strength is

$$\cdot\, m_n = \cdot\, Rbd^2 = 0.90 \times 124 \times 12 \times 5.5^2 \cdot 12 = 3376 \text{ ft-lb} \cdot \text{ft}$$

In comparison with the required strengths summarized in Fig. 15.17*d*, this will be adequate everywhere except for particular regions as follows:

Negative steel over column:

$$R = \frac{m_u}{\cdot\, bd^2} = \frac{10{,}528 \times 12}{0.90 \times 12 \times 5.25^2} = 424$$

for which $\cdot = 0.0076$ (from Table A.5a), and $A_s = 0.0076 \times 12 \times 5.25 = 0.48$ in^2/ft. This will be provided using No. 5 (No. 16) bars at 7.5 in. spacing. They will be continued a distance $0.6 \times 9.11 = 5.47$ ft, say 5 ft 6 in., to the left of the column centerline, and $0.6 \times 10.32 = 6.19$ ft, say 6 ft 3 in., to the right.

Negative steel along fixed edges:

$$R = \frac{3509 \times 12}{0.90 \times 12 \times 5.50^2} = 129$$

for which $\cdot = 0.0022$ and $A_s = 0.0022 \times 12 \times 5.5 = 0.15$ in^2/ft. No. 3 (No. 10) bars at 9 in. spacing will be adequate. The point of inflection for the slab in this region is easily found to be 3.30 ft from the fixed edge. The negative bars will be extended 5.5 in. beyond that point, resulting in a cutoff 45 in., or 3 ft 9 in., from the support face.

Positive steel in outer spans:

$$R = \frac{3789 \times 12}{0.90 \times 12 \times 5.50^2} = 139$$

resulting in $\cdot = 0.0024$ and $A_s = 0.0024 \times 12 \times 5.5 = 0.16$ in^2/ft. No. 3 (No. 10) bars at 8 in. spacing will be used. In all cases, the maximum spacing of $2h = 13$ in. is satisfied. That maximum would preclude the economical use of larger diameter bars.

Bar size and spacing and cutoff points for the top and bottom steel are summarized in Fig. 15.17*e* and *f*, respectively.

Finally, the load carried by the central column is

$$P = 170 \times 19.43 \times 19.43 = 64{,}200 \text{ lb}$$

Investigating punching shear at a critical section taken $d/2$ from the face of the 16 in. column, with reference to Eq. (13.11*a*) and with $b_o = 4 \times (16.00 + 5.25) = 85$ in., gives

$$\phi V_c = 4\phi \sqrt{f'_c}\, b_o d = 4 \times 0.75 \sqrt{4000} \times 85 \times 5.25 = 84{,}700 \text{ lb}$$

This is well above the applied shear of 64,200 lb, confirming that the slab thickness is adequate and that no shear reinforcement is required.

15.9 Comparisons of Methods for Slab Analysis and Design

The conventional methods of slab analysis and design, as described in Chapter 13 and as treated in Chapter 13 of the ACI Code, are limited to applications in which slab panels are supported on opposite sides or on all four sides by beams or walls or to the case of flat plates and related forms supported by a relatively regular array of columns. In all cases, slab panels must be square or rectangular, loads must be uniformly distributed within each panel, and slabs must be free of significant holes.

Both the yield line theory and the strip method offer the designer rational methods for slab analysis and design over a much broader range, including the following:

1. Boundaries of any shape, including rectangular, triangular, circular, and L-shaped boundaries with reentrant corners
2. Supported or unsupported edges, skewed supports, column supports, or various combinations of these conditions
3. Uniformly distributed loads, loads distributed over partial panel areas, linear varying distributed loads, line loads, and concentrated loads
4. Slabs with significant holes

The most important difference between the strip method and the yield line method is the fact that the strip method produces results that are always on the safe side, but yield line analysis may result in unsafe designs. A slab designed by the strip method may possibly carry a higher load than estimated, through internal force redistributions, before collapse; a slab analyzed by yield line procedures may fail at a lower load than anticipated if an incorrect mechanism has been selected as the basis or if the defining dimensions are incorrect.

Beyond this, it should be realized that the strip method is a tool for *design*, by which the slab thickness and reinforcing bar size and distribution may be selected to resist the specified loads. In contrast, the yield line theory offers only a means for *analyzing the capacity of a given slab*, with known reinforcement. According to the yield line approach, the design process is actually a matter of reviewing the capacities of a number of trial designs and alternative reinforcing patterns. All possible yield line patterns must be investigated and specific dimensions varied to be sure that the correct solution has been found. Except for simple cases, this is likely to be a time-consuming process.

Neither the strip method nor the yield line approach provides any information regarding cracking or deflections at service load. Both focus attention strictly on flexural strength. However, by the strip method, if care is taken at least to approximate the elastic distribution of moments, little difficulty should be experienced with excessive cracking. The methods for deflection prediction presented in Section 13.13 can, without difficulty, be adapted for use with the strip method, because the concepts are fully compatible.

With regard to economy of reinforcement, it might be supposed that use of the strip method, which always leads to designs on the safe side, might result in more expensive structures than the yield line theory. Comparisons, however, indicate that, in most cases, this is not so (Refs. 15.8 and 15.12). Through proper use of the strip method, reinforcing bars are placed in a nonuniform way in the slab (e.g., in strong bands around openings) where they are used to best effect; yield line methods, on the other hand, often lead to uniform bar spacings, which may mean that individual bars are used inefficiently.

Many tests have been conducted on slabs designed by the strip method (Ref. 15.11; also, see the summary in Ref. 15.12). These tests included square slabs, rectangular slabs, slabs with both fixed and simply supported edges, slabs supported directly by columns, and slabs with large openings. The conclusions drawn determine that the strip method provides for safe design with respect to nominal strength and that at service load, behavior with respect to cracking and deflections is generally satisfactory. The method has been widely and successfully used in the Scandinavian countries since the 1960s.

15.1. A. Hillerborg, "Equilibrium Theory for Reinforced Concrete Slabs" (in Swedish), *Betong*, vol. 41, no. 4, 1956, pp. 171–182.

15.2. A. Hillerborg, *Strip Method for Slabs on Columns, L-Shaped Plates, Etc.* (in Swedish), Svenska Riksbyggen, Stockholm, 1959.

15.3. A. Hillerborg, "A Plastic Theory for the Design of Reinforced Concrete Slabs," *Proc. Sixth Congr.*, International Association for Bridge and Structural Engineering, Stockholm, 1960.

15.4. R. E. Crawford, *Limit Design of Reinforced Concrete Slabs*, thesis submitted to University of Illinois for the degree of Ph.D., Urbana, IL, 1962.

15.5. F. A. Blakey, *Strip Method for Slabs on Columns, L-Shaped Plates, Etc.* (translation of Ref. 15.2), Commonwealth Scientific and Industrial Research Organization, Melbourne, 1964.

15.6. K. O. Kemp, "A Lower Bound Solution to the Collapse of an Orthotropically Reinforced Slab on Simple Supports," *Mag. Concr. Res.*, vol. 14, no. 41, 1962, pp. 79–84.

15.7. K. O. Kemp, "Continuity Conditions in the Strip Method of Slab Design," *Proc. Inst. Civ. Eng.*, vol. 45, 1970, p. 283 (supplement paper 7268s).

15.8. R. H. Wood and G. S. T. Armer, "The Theory of the Strip Method for the Design of Slabs," *Proc. Inst. Civ. Eng.*, vol. 41, 1968, pp. 285–311.

15.9. R. H. Wood, "The Reinforcement of Slabs in Accordance with a Predetermined Field of Moments," *Concrete*, vol. 2, no. 2, 1968, pp. 69–76.

15.10. G. S. T. Armer, "The Strip Method: A New Approach to the Design of Slabs," *Concrete*, vol. 2, no. 9, 1968, pp. 358–363.

15.11. G. S. T. Armer, "Ultimate Load Tests of Slabs Designed by the Strip Method," *Proc. Inst. Civ. Eng.*, vol. 41, 1968, pp. 313–331.

15.12. A. Hillerborg, *Strip Method of Design*, Viewpoint Publications, Cement and Concrete Association, Wexham Springs, Slough, England, 1975.

15.13. A. Hillerborg, "The Advanced Strip Method—a Simple Design Tool," *Mag. Concr. Res.*, vol. 34, no. 121, 1982, pp. 175–181.

15.14. R. Park and W. L. Gamble, *Reinforced Concrete Slabs*, 2nd ed, (Chapter 6), John Wiley, New York, 2000, pp. 232–302.

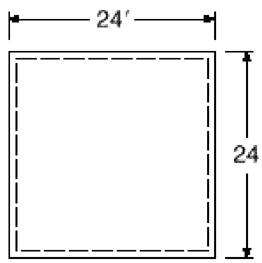
15.15. A. Hillerborg, *Strip Method Design Handbook*, E & FN Spon/Chapman & Hill, London, 1996.

15.16. S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, 2nd ed., McGraw-Hill, New York, 1959.

Note: For all the following problems, use material strengths $f_y = 60{,}000$ psi and $f'_c = 4000$ psi. All ACI Code requirements for minimum steel, maximum spacings, bar cutoff, and special corner reinforcement are applicable.

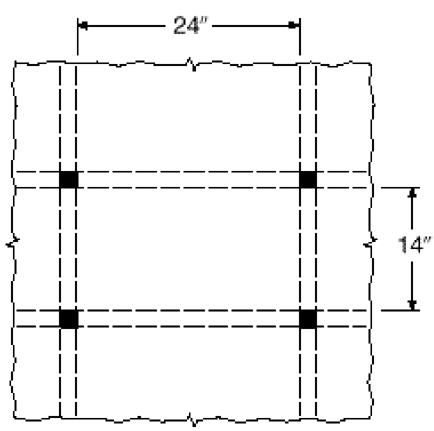
15.1. The square slab of Fig. P15.1 is simply supported by masonry walls along all four sides. It is to carry a service live load of 100 psf in addition to its self-weight. Specify a suitable load distribution; determine moments at all controlling sections; and select the slab thickness, reinforcing bars, and spacing.

FIGURE P15.1



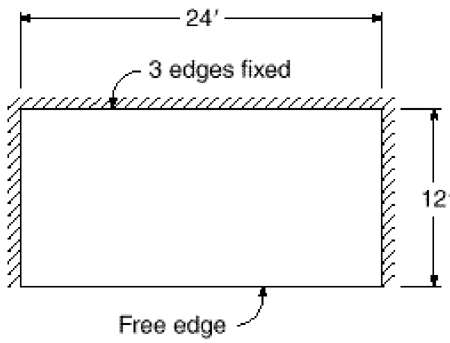
15.2. The rectangular slab shown in Fig. P15.2 is a typical interior panel of a large floor system having beams on all column lines. Columns and beams are sufficiently stiff that the slab can be considered fully restrained along all sides. A live load of 100 psf and a superimposed dead load of 30 psf must be carried in addition to the slab self-weight. Determine the required slab thickness, and specify all reinforcing bars and spacings. Cutoff points for negative bars should be specified; all positive steel may be carried into the supporting beams. Take support moments to be 2 times the span moments in the strips.

FIGURE P15.2



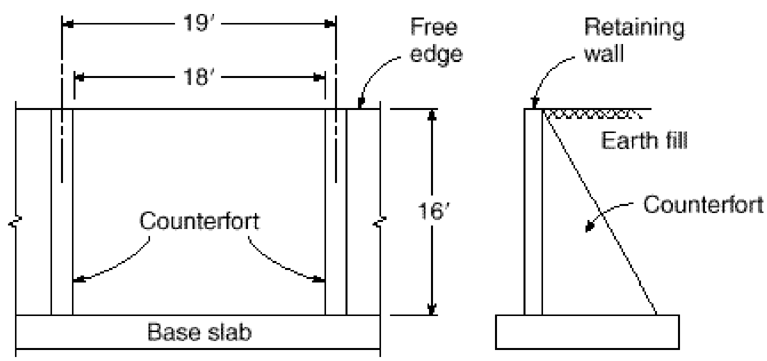
15.3. The slab of Fig. P15.3 may be considered fully fixed along three edges, but it is without support along the fourth, long side. It must carry a uniformly distributed live load of 80 psf plus an external dead load of 40 psf. Specify a suitable slab depth, and determine reinforcement and cutoff points.

FIGURE P15.3



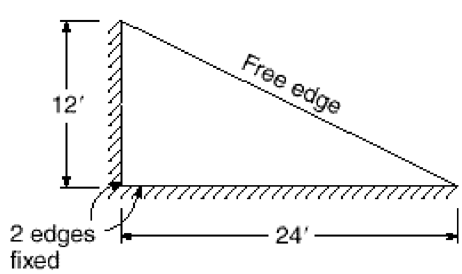
15.4. Figure P15.4 shows a counterfort retaining wall (see Section 17.9) consisting of a base slab and a main vertical wall of constant thickness retaining the earth. Counterfort walls spaced at 19 ft on centers along the wall provide additional support for the main slab. Each section of the main wall, which is 16 ft high and 18 ft long, may be considered fully fixed at its base and also along its two vertical sides (because of full continuity and identical loadings on all such panels). The top of the main wall is without support. The horizontal earth pressure varies from 0 at the top of the wall to 587 psf at the top of the base slab. Determine a suitable thickness for the main wall, and select reinforcing bars and spacing.

FIGURE P15.4



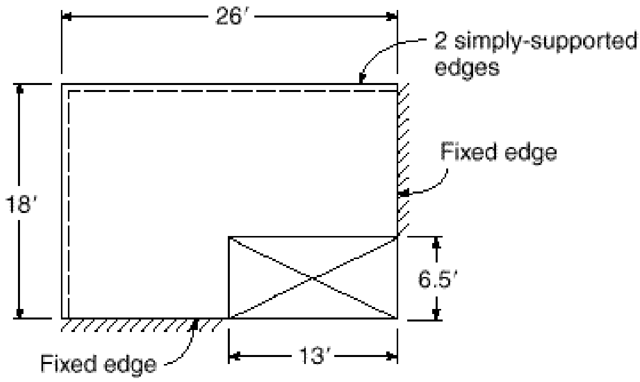
15.5. The triangular slab shown in Fig. P15.5, providing cover over a loading dock, is fully fixed along two adjacent sides and free of support along the diagonal edge. A uniform snow load of 60 psf is anticipated. Dead load of 10 psf will act, in addition to self-weight. Determine the required slab depth and specify all reinforcement. (Hint: The main bottom reinforcement should be parallel to the free edge, and the negative reinforcement should be perpendicular to the supported edges.)

FIGURE P15.5



15.6. Figure P15.6 shows a rectangular slab with a large opening near one corner. It is simply supported along one long side and the adjacent short side, and the two edges adjacent to the opening are fully fixed. A factored load of 250 psf must be carried. Find the required slab thickness, and specify all reinforcement.

FIGURE P15.6



15.7. The roof deck slab of Fig. P15.7 is intended to carry a total factored load, including self-weight, of 165 psf. It will have fixed supports along the two long sides and one short side, but the fourth edge must be free of any support. Two 16 in. square columns will be located as shown.
(*a*) Determine an acceptable slab thickness.
(*b*) Select appropriate load dispersion lines.
(*c*) Determine moments at all critical sections.
(*d*) Specify bar sizes, spacings, and cutoff points.
(*e*) Check controlling sections in the slab for shear strength.

FIGURE P15.7